WKB solution of the wave equation for a plane angular sector

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The two angular Lamé differential equations that satisfy boundary conditions on a plane angular sector (PAS) are solved by the Wentzel-Kramers-Brillouin (WKB) method. The WKB phase constants are derived by matching the WKB solution with the asymptotic solution of the Weber equation. The WKB eigenvalues and eigenfunctions show excellent agreement with the exact eigenvalues and eigenfunctions. It is shown that those WKB eigenvalues and eigenfunctions that contribute substantially to the scattering amplitude from a PAS can be computed in a rather simple way. An approximate formula for the WKB normalization constant, which is consistent with the WKB assumptions, is derived and compared with the exact normalization constant.

I. INTRODUCTION

In a previous paper [1], formulas for the Wentzel-Kramers-Brillouin (WKB) eigenvalues satisfying Dirichlet or Neumann boundary condition on a plane angular sector (PAS) were reported and the WKB eigenvalues, \( \nu, \mu \), were compared with the exact eigenvalues for PAS’s of different corner angles (60°, 90°, and 120°). A historical review of the solution of the wave equation for a PAS was also given in the above paper. It suffices to say that, to our knowledge, no approximate solution of this problem has been reported in the literature.

In this paper, the two coupled Lamé equations are solved by the WKB method. The WKB analysis in this paper is valid for large values of \( \nu \). Depending on the sign of \( \mu \), one of the two Lamé equations has turning points. When \( |\mu| \) is small the turning points occur where the angles are small. In this region it proves more accurate to obtain the WKB phase constants by matching it with the asymptotic solution of the Weber equation. For large values of \( |\mu| \), it is shown that this phase constant reduces to the phase constant obtained when the solution is matched with the asymptotic solution of the Airy’s equation as is commonly done in quantum mechanics.

This paper is organized in the following way: In the second section the WKB solution is formulated and the WKB phase constants are derived. In the third section formulas for the WKB eigenvalues for Dirichlet and Neumann boundary conditions are derived and a comparison between the WKB and exact eigenfunctions is presented. The fourth section contains a derivation of approximate WKB solutions which are valid for small values of \( \mu/\nu \). Finally, an approximate formula for the WKB normalization constant consistent with WKB assumptions is derived in section five.

II. THE GENERAL SOLUTION

The angular part of the wave equation in the spherico-conal coordinate system are expressed by the two angular Lamé differential equations [2,1].

\[
\sqrt{1 - \kappa^2 \cos^2 \vartheta} \frac{d}{d\vartheta} \left[ \sqrt{1 - \kappa' \cos^2 \vartheta} \frac{d}{d\vartheta} \Theta(\vartheta) \right] + [\nu (\nu + 1) \kappa^2 \sin^2 \vartheta + \mu] \Theta(\vartheta) = 0
\]

and

\[
\sqrt{1 - \kappa'^2 \sin^2 \varphi} \frac{d}{d\varphi} \left[ \sqrt{1 - \kappa^2 \cos^2 \varphi} \frac{d}{d\varphi} \Phi(\varphi) \right] + [\nu (\nu + 1) \kappa^2 \sin^2 \varphi - \mu] \Phi(\varphi) = 0,
\]

where the spherico-conal coordinate system variables \( \vartheta \) and \( \varphi \) are related to the Cartesian coordinate variables, \( x \), \( y \), and \( z \) by

\[
x = r \cos \vartheta \sqrt{1 - \kappa'^2 \cos^2 \varphi},
\]

\[
y = r \sin \vartheta \sin \varphi,
\]

\[
z = r \cos \varphi \sqrt{1 - \kappa^2 \cos^2 \vartheta},
\]

where

\[
\kappa = \cos \left( \frac{\beta}{2} \right),
\]

\( \beta \) is the corner angle and \( \kappa' = \sqrt{1 - \kappa^2} \). The range of the variables is

\[0 \leq \vartheta \leq \pi, \quad 0 \leq \varphi \leq 2 \pi, \quad r \geq 0.\]

The geometry of the spherico-conal coordinate system is shown in Fig. 1. The construction of this coordinate system is described, and its orthogonality proved, in [2]. The WKB solution of Eqs. (1) and (2) are respectively given by [1]
is the distance from the origin to the point $p$.

$$\Theta(\vartheta) = \frac{1}{\sqrt[4]{(\nu + \frac{1}{2})^2 \kappa^2 \sin^2 \vartheta + \mu}} \times \cos \left( \int_0^{\vartheta} \left( \frac{(\nu + \frac{1}{2})^2 \kappa^2 \sin^2 \vartheta + \mu}{1 - \kappa^2 \cos^2 \vartheta} \right)^{1/2} d \vartheta + \delta_\Theta \right),$$

(4)

and

$$\Phi(\varphi) = \frac{1}{\sqrt[4]{(\nu + \frac{1}{2})^2 \kappa^2 \sin^2 \varphi - \mu}} \times \cos \left( \int_0^{\varphi} \left( \frac{(\nu + \frac{1}{2})^2 \kappa^2 \sin^2 \varphi - \mu}{1 - \kappa^2 \cos^2 \varphi} \right)^{1/2} d \varphi + \delta_\Phi \right).$$

(5)

Notice that the $\Phi$ solution can be obtained from the $\Theta$ solution by replacing $\vartheta$ with $\varphi$, $\mu$ with $-\mu$, and $\kappa$ with $\kappa'$. This is also evident in the differential equations, Eqs. (1) and (2). For $\mu>0$, the turning point for the $\Theta$ equation, $\vartheta_1=0$, and the turning point for the $\Phi$ equation is $\varphi_1=\theta_1$, $\vartheta_1=\varphi_1$. For $\mu<0$, $\varphi_1=0$ and $\theta_1=\theta_1(\nu, \mu, \kappa)$, where

$$\theta_1 = \arcsin \left( \frac{-\mu}{\kappa^2 (\nu + \frac{1}{2})^2} \right)^{1/2}.$$

The above solutions are valid for $\pi - \vartheta_1 > \vartheta > \vartheta_1$ and $\pi - \varphi_1 > \varphi > \varphi_1$. For the regions $\vartheta > \vartheta_1 > \pi - \vartheta_1$ and $\varphi > \varphi_1 > \pi - \varphi_1$, the solutions are given by

$$\Theta(\vartheta) = \frac{1}{\sqrt[4]{\left| \mu - (\nu + \frac{1}{2})^2 \kappa^2 \sin^2 \vartheta \right|}} \times \exp \left( - \int_0^{\vartheta} \left( \frac{\left| \mu - (\nu + \frac{1}{2})^2 \kappa^2 \sin^2 \vartheta \right|}{1 - \kappa^2 \cos^2 \vartheta} \right)^{1/2} d \vartheta \right),$$

(6)

for $\mu<0$. A similar solution for the $\Phi$ equation which is valid for $\mu>0$ can be obtained from the above equation by replacing $|\mu|$, $\kappa$, and $\vartheta$ with $\mu$, $\kappa'$, and $\varphi$, respectively. The solutions given by Eqs. (4) and (6) are separated by the turning points where they are both singular. To match these solutions at the turning points, the Lamé equation is approximated by another differential equation at the turning point and then the asymptotic solution of this differential equation is matched with the WKB solution of the Lamé equation. By this process the phase constants $\delta_\Theta$ and $\delta_\Phi$ are obtained.

Although the WKB solutions obtained in this paper are general, we are particularly interested in those solutions that correspond to the small absolute values of the eigenvalue $\mu$. The reason for this is that the expression for the scattering amplitude for a PAS, derived in a separate paper [3], contains either the eigenfunctions or their derivatives evaluated at the surface of the PAS or its edges. Observe that the surface of the PAS in the sphero-conal coordinate system is $\vartheta=\pi$ and its edges are $\varphi=0$ and $\varphi=\pi$. Since the WKB eigenfunctions are decaying exponentials to the left of the turning point near zero and to the right of the turning point near $\pi$, significant contribution to the solution comes from those eigenfunctions that have a turning point near the surface or near the edges. They correspond to eigenfunctions with small absolute values of the eigenvalue $\mu$.

To derive a differential equation that approximates the Lamé equation at the turning points, we use the following transformation:

$$u(\vartheta) = \frac{1}{\sqrt[4]{1 - \kappa^2 \cos^2(\vartheta)}} \Theta(\vartheta).$$

For $\nu \gg \mu \gg 1$ this transformation converts Eq. (1) to

$$\frac{d^2}{d\vartheta^2} u(\vartheta) + p(\vartheta) u(\vartheta) = 0,$$

(7)

with

$$p(\vartheta) = \frac{(\nu + \frac{1}{2})^2 \kappa^2 \sin^2 \vartheta + \mu}{(1 - \kappa^2 \cos^2 \vartheta)}.$$  

By using the Liouville transformation [4]

$$y(x) = \left( \frac{dx}{d\vartheta} \right)^{1/2} u(\vartheta),$$

Eq. (7) can be transformed to

$$\frac{d^2 y}{dx^2} = \left[ - \left( \frac{d \vartheta}{dx} \right)^2 p(\vartheta) + \left( \frac{d \vartheta}{dx} \right)^{1/2} \frac{d^2}{d\vartheta^2} \left[ \left( \frac{d \vartheta}{dx} \right)^{1/2} \right] \right] y.$$  

The first term in the curly brackets can be set equal to any smooth function of $x$ [4], and for small $\mu/\nu$ the second term can be ignored. Near the turning points ($\vartheta<\pi$) we set

$$\left( \frac{d \vartheta}{dx} \right)^2 p(\vartheta) = \left( \frac{x^2}{4} + a \right),$$

(8)

resulting in

$$\frac{d^2 y}{dx^2} + \left( \frac{x^2}{4} + a \right) y = 0,$$

(9)
which is the Weber equation. In the above equation the plus sign is applicable to the \( \Theta \) equation and the minus sign is applicable to the \( \Phi \) equation. The parameter \( a \) is determined from Eq. (8) by requiring that the turning points of the Lamé equation and the Weber equation occur at the same time, thus ensuring the regularity of \( d\vartheta/dx \) at the turning points:

\[
\int_0^{\vartheta} \left( \frac{(v^2 + \frac{1}{2})\kappa^2 \sin^2 \vartheta + \mu}{1 - \kappa^2 \cos^2 \vartheta} \right)^{1/2} \, d\vartheta = \int_0^{\pi} \left( \frac{x^2}{4} + a \right)^{1/2} \, dx.
\]

(10)

A similar equation is obtained for the \( \Phi \) equation by replacing \( \vartheta \) by \( \varphi \), \( \kappa \) by \( \kappa' \), \( \mu \) by \( -\mu \), and \( a \) by \( -a \). Once \( a \) is determined from the above equation, \( \vartheta \) and \( \varphi \) can be related to \( x \) by

\[
\int_0^{\vartheta} \left( \frac{(v^2 + \frac{1}{2})\kappa^2 \sin^2 \vartheta + \mu}{1 - \kappa^2 \cos^2 \vartheta} \right)^{1/2} \, d\vartheta = \int_0^{\pi} \left( \frac{x^2}{4} + a \right)^{1/2} \, dx.
\]

(11)

and

\[
\int_0^{\varphi} \left( \frac{(v^2 + \frac{1}{2})\kappa'^2 \sin^2 \varphi - \mu}{1 - \kappa'^2 \cos^2 \varphi} \right)^{1/2} \, d\varphi = \int_{2\pi}^{\pi} \left( \frac{x^2}{4} - a \right)^{1/2} \, dx.
\]

(12)

\( a \) is found to be

\[
a = \frac{2}{\pi} \left\{ \frac{(v^2 + \frac{1}{2})\kappa^2 - \mu}{\sqrt{(v^2 + \frac{1}{2})\kappa^2 + \mu}} \right\} \times \left[ \Pi \left( \frac{\pi}{2}, \frac{\mu}{(v^2 + \frac{1}{2})\kappa^2 + \mu} \right) - K(e) \right].
\]

(13)

where

\[
e = \frac{1}{\kappa'} \left( \frac{\mu}{(v^2 + \frac{1}{2})\kappa^2 + \mu} \right)^{1/2}, \quad \mu > 0,
\]

and \( \Pi \) is the elliptic integral of the third kind. For \( \mu < 0 \) \( a \) can be obtained from the above equation by replacing \( \mu \) by \( |\mu| \) and interchanging \( \kappa' \) and \( \kappa \). Equation (13) has a power series expansion given by

\[
a = \frac{1}{2\kappa \kappa'} \chi + \frac{\kappa^2 - \kappa'^2}{16\kappa^3 \kappa'^3 (v^2 + \frac{1}{2})} \chi^3 + \frac{3 - 8\kappa^2 \kappa'^2}{128\kappa^5 \kappa'^5 (v^2 + \frac{1}{2})} \chi^3 + O(\chi^4),
\]

where

\[
\chi = \frac{\mu}{v^2 + \frac{1}{2}}.
\]

If only the first term in the above expansion is retained,

\[
a = \frac{\mu}{2\kappa \kappa'} \left( v^2 + \frac{1}{2} \right).
\]

(14)

If this value of \( a \) is used in Eqs. (11) and (12), near the turning points (small \( \vartheta \) and \( \varphi \)) the above equations respectively yield

\[
x = \sqrt{2\kappa' \kappa} \sqrt{(v^2 + \frac{1}{2}) \vartheta}, \quad x = \sqrt{2\kappa' \kappa} \sqrt{(v^2 + \frac{1}{2}) \varphi}.
\]

(15)

Equation (9) is the desired differential equation. The asymptotic solution of this equation will be used to determine \( \delta_\vartheta \) and \( \delta_\varphi \).

A. The WKB phase constants \( \delta_\vartheta \) and \( \delta_\varphi \)

The Weber equation

\[
\frac{d^2y}{dx^2} + \left( \frac{x^2}{4} - a \right)y = 0
\]

has solutions [5]

\[
W(a, \pm x) = 2^{-3i\delta}(\sqrt{G_1/G_3}y_c + \sqrt{2G_3/G_1}y_o),
\]

(17)

where

\[
y_c(x, a) = 1 + \frac{x^2}{2!} + \left( \frac{x^2}{2} \right)^2 \frac{1}{4!} \cdots
\]

(18)

\[
y_c(x, a) = x + \frac{x^3}{3!} + \left( \frac{x^3}{3} \right)^2 \frac{1}{5!} \cdots
\]

(19)

and

\[
G_1(a) = \Gamma \left( \frac{1}{4} + \frac{1}{2}ia \right),
\]

\[
G_3(a) = \Gamma \left( \frac{1}{4} + \frac{1}{2}ia \right).
\]

For \( x \gg |a| \) the Weber equation has asymptotic solutions [5]

\[
W(a, x) = \left( \frac{2\eta(a)}{x} \right)^{1/2} \cos \left( \frac{1}{4} x^2 - a \ln(x) + \frac{\pi}{4} + \frac{1}{2} \phi_2(a) \right)
\]

and

\[
W(a, -x) = \left( \frac{2}{\eta(a)x} \right)^{1/2} \sin \left( \frac{1}{4} x^2 - a \ln(x) + \frac{\pi}{4} + \frac{1}{2} \phi_2(a) \right),
\]

where

\[
\eta(a) = \sqrt{1 + e^{2\pi a} - e^{2\pi a}}, \quad \phi_2(a) = \arg \Gamma \left( \frac{1}{2} + ia \right).
\]

From these two solutions we construct an even solution
In the region where only terms of first order gives

\[
\tilde{y}_e(x,a) = \frac{1}{2} \left( \frac{G_3(a)}{G_1(a)} \right)^{1/2} \left( 1 + \eta^2(a) \right)^{1/2} \left( \frac{1}{\eta(a)} \right)^{1/2} \times \cos \left( \frac{x^2}{4} - a \ln x + \frac{\pi}{4} + \frac{1}{2} \phi_2(a) - \gamma(a) \right),
\]

and an odd solution

\[
\tilde{y}_o(x,a) = -\frac{1}{2} \sqrt{\frac{G_1(a)}{2G_3(a)}} \left( 1 + \eta^2(a) \right)^{1/2} \left( \frac{1}{\eta(a)} \right)^{1/2} \times \cos \left( \frac{x^2}{4} - a \ln x + \frac{\pi}{4} + \frac{1}{2} \phi_2(a) + \gamma(a) \right),
\]

where

\[
\tan \gamma(a) = \frac{1}{\eta(a)} = \sqrt{1 + e^{\pi a} e^{\pi a}}.
\]

Comparing the phase and amplitude of this equation with that of \(\tilde{y}_e(x,a)\) we find

\[
\delta_\varphi = -\frac{\pi}{8} + \frac{1}{2} \phi_2(a) - \frac{1}{2} D(a) - \frac{a}{2} \ln |a| + \frac{a}{2}
\]

and

\[
A_\varphi = 4 \sqrt{k \kappa' \left(\nu + \frac{1}{2}\right)} \left( \frac{G_3(a)}{G_1(a)} \right)^{1/2} \left( 1 + \eta^2(a) \right)^{1/2} \left( \frac{1}{\eta(a)} \right)^{1/2}.
\]

It can be shown that

\[
2 \gamma(a) - \frac{3}{4} \pi = \arctan \left( \frac{\pi a}{2} \right) = D(a) - D(-a).
\]

In terms of this new quantity we have

\[
\delta_\varphi = \frac{5 \pi}{8} + \frac{1}{2} \phi_2(-a) + \frac{1}{2} D(-a) - \frac{a}{2} \ln |a| + \frac{a}{2}.
\]

The phase constant for the odd solution is obtained by comparing \(I_\varphi\) with the phase of Eq. (21). It is

\[
\delta_\theta = -\frac{\pi}{8} - \frac{1}{2} D(-a) + \frac{a}{2} |\ln a| - \frac{a}{2}.
\]

A similar analysis gives

\[
\delta_\theta = -\frac{\pi}{8} + \frac{1}{2} D(-a) - \frac{a}{2} |\ln a| - \frac{a}{2}.
\]

And for the odd solution we find

\[
\delta_\theta = \frac{5 \pi}{8} - \frac{1}{2} \phi_2(-a) - \frac{1}{2} D(a) + \frac{a}{2} |\ln a| - \frac{a}{2}.
\]

For \(\mu < 0\) the role of the \(\Phi\) and the \(\Theta\) equations are interchanged. In other words, in this case the \(\Theta\) equation is the one with the turning points. However, the expressions obtained for the phase constants still remain valid. In summary we have

\[
\delta_\varphi = -\frac{\pi}{8} - \frac{1}{2} \phi_2(a) - \frac{1}{2} D(a) - \frac{a}{2} |\ln a| + \frac{a}{2},
\]

\[
\delta_\theta = -\frac{\pi}{8} - \frac{1}{2} \phi_2(a) - \frac{1}{2} D(a) + \frac{a}{2} |\ln a| - \frac{a}{2},
\]

\[
\delta_\varphi' = \frac{5 \pi}{8} - \frac{1}{2} \phi_2(a) + \frac{1}{2} D(a) - \frac{a}{2} |\ln a| + \frac{a}{2},
\]

\[
\delta_\theta' = \frac{5 \pi}{8} - \frac{1}{2} \phi_2(a) + \frac{1}{2} D(a) + \frac{a}{2} |\ln a| - \frac{a}{2}.
\]
The above phase constants were derived for small values of $a$ when the turning points lie close to 0 or $\pi$. For large positive values of $a$, $D(a)\to \pi/4$ and $F(a)\to \ln|a|\pm a$. In this limit $\delta_\phi = -\pi/4$, $\delta_\theta = 3\pi/2$, $\delta_\phi = 0$, and $\delta_\theta = \pi/2$. For large negative values of $a$, $D(a)\to -\pi/4$ and $F(a)\to \ln(-a)\pm a$. In this limit $\delta_\phi = 0$, $\delta_\theta = \pi/2$, $\delta_\phi = -\pi/4$, and $\delta_\theta = 3\pi/2$. These are the phase constants that would have been obtained had the problem been treated by the regular WKB method often employed in solving the Schrödinger equation in quantum mechanics. In solving the Schrödinger equation by the WKB method, the phase constants are obtained by matching the phase of the WKB solution with the phase of the asymptotic solution of the Airy’s equation.

III. THE WKB SOLUTIONS

By determining the phase terms, $\delta_\phi$, $\delta_\theta$, $\delta_\phi'$, and $\delta_\theta'$ we now have the complete WKB solutions of the $\Theta$ and the $\Phi$ equations. In this section we derive the WKB eigenvalue equations by applying the boundary conditions and imposing the requirement that the solutions for $\Theta(\theta;\varphi) > (\partial_\varphi \varphi)$ and those for $\Theta(\theta;\varphi) < (\partial_\varphi \varphi)$ join each other smoothly in their common region of validity. This results in relationships for the WKB eigenvalues. Both Dirichlet and Neumann boundary conditions will be considered.

A. Dirichlet boundary condition

For the Dirichlet boundary condition we have [1]

$$\Phi'(0) = 0, \quad \Theta'(0) = 0,$$

$$\Phi'(\pi) = 0, \quad \Theta(\pi) = 0.$$

That is, the $\Phi$ solution must be even at both $\varphi=0$ and $\varphi=\pi$, where the $\Theta$ solution must be even at $\vartheta=0$ and odd at $\vartheta=\pi$. The WKB solution of the $\Phi$ equation valid in the region $\varphi > \varphi_1$, is

$$\Phi^<(\varphi) = A_\varphi h(\varphi) \cos \left( \int_{\varphi_1}^{\varphi} \frac{(v + \frac{1}{2})^2 \kappa^2 \sin^2 \varphi - \mu}{1 - \kappa^2 \cos^2 \varphi} \, d\varphi + \delta_\varphi \right), \quad \varphi > \varphi_1,$$

where

$$h(\varphi) = \frac{1}{\sqrt{\left( v + \frac{1}{2} \right)^2 \kappa^2 \sin^2 \varphi - \mu}}.$$ 

Since for the Dirichlet boundary condition $\Phi(\varphi)$ is even at both $\varphi=0$ and $\varphi=\pi$, the WKB solution for $\varphi < \pi - \varphi_1$ is given by

$$\Phi^>(\varphi) = A_\varphi h(\varphi) \cos \left( \int_{\varphi_1}^{\pi - \varphi_1} \frac{(v + \frac{1}{2})^2 \kappa^2 \sin^2 \varphi - \mu}{1 - \kappa^2 \cos^2 \varphi} \, d\varphi + \delta_\varphi \right), \quad \varphi < \pi - \varphi_1.$$

The two solutions, $\Phi^<(\varphi)$ and $\Phi^>(\varphi)$, must join smoothly in the region $\varphi_1 < \varphi < \pi - \varphi_1$. Let us define

$$J_\varphi = \int_{\varphi_1}^{\pi - \varphi_1} \frac{(v + \frac{1}{2})^2 \kappa^2 \sin^2 \varphi - \mu}{1 - \kappa^2 \cos^2 \varphi} \, d\varphi$$

$$= 2 \int_{\varphi_1}^{\pi/2} \frac{(v + \frac{1}{2})^2 \kappa^2 \sin^2 \varphi - \mu}{1 - \kappa^2 \cos^2 \varphi} \, d\varphi, \quad (26)$$

and

$$b = \int_{\varphi}^{\pi - \varphi_1} \frac{(v + \frac{1}{2})^2 \kappa^2 \sin^2 \varphi - \mu}{1 - \kappa^2 \cos^2 \varphi} \, d\varphi + \delta_\varphi.$$ 

In terms of these quantities we have

$$\Phi^<(\varphi) = A_\varphi h(\varphi) \cos (J_\varphi - b + 2\delta_\varphi)$$

and

$$\Phi^>(\varphi) = A_\varphi h(\varphi) \cos b.$$ 

Equating $\Phi^<$ and $\Phi^>$ gives
A_e \{ \cos(\varphi + 2 \delta_e) \cos b + \sin(\varphi + 2 \delta_e) \sin b \} = A'_e \cos b.

The solution to this equation, which gives \( A'_e \), constant and independent of the parameter \( b \), is obtained by setting

\[ J^2 + 2 \delta_e = m \pi, \quad m = 0, 1, \ldots. \]

Substituting for \( \delta_e \), we find

\[ J^2 + \phi_2(a) - D(a) - a \ln |a| + a = (m + \frac{1}{2}) \pi, \quad m = 0, 1, \ldots. \]

For Dirichlet boundary condition \( \Theta(\vartheta) \) is even around \( \vartheta = 0 \) and odd around \( \vartheta = \pi \). Then

\[ \Theta^<(\vartheta) = A_0 h(\vartheta) \cos \left( \int_0^\vartheta \frac{(\nu + \frac{1}{2})^2 \kappa^2 \sin^2 \vartheta + \mu}{1 - \kappa^2 \cos^2 \vartheta} d \vartheta + \delta_0 \right), \quad \vartheta > 0, \]

where \( h(\vartheta) \) is \( h(\varphi) \) with the appropriate change of variables. The solution for \( \vartheta > \pi + \vartheta_1 \) is

\[ \Theta^>(\vartheta) = A'_0 h(\vartheta) \cos \left( \int_{\pi + \vartheta_1}^\vartheta \frac{(\nu + \frac{1}{2})^2 \kappa^2 \sin^2 \vartheta + \mu}{1 - \kappa^2 \cos^2 \vartheta} d \vartheta + \delta'_0 \right), \quad \vartheta > \pi + \vartheta_1. \]

Since this solution is odd with respect to \( \vartheta = \pi \), the solution for \( \vartheta < \pi - \vartheta_1 \) is given by

\[ \Theta^>(\vartheta) = -A'_0 h(\vartheta) \cos \left( \int_0^{\vartheta - \vartheta_1} \frac{(\nu + \frac{1}{2})^2 \kappa^2 \sin^2 \vartheta + \mu}{1 - \kappa^2 \cos^2 \vartheta} d \vartheta + \delta'_0 \right), \quad \vartheta < \pi - \vartheta_1. \]

Requiring that \( \Theta^<(\vartheta) \) and \( \Theta^>(\vartheta) \) join each other smoothly results in

\[ J_\vartheta + \delta_\vartheta + \delta'_\vartheta = n \pi, \quad n = 1, 2, \ldots. \]

Note that the reason \( n \) starts from 1 is to guarantee that

\[ n \pi - \delta_\vartheta - \delta'_\vartheta > 0, \]

since \( J_\vartheta \) is positive. Substituting for \( \delta_\vartheta \) and \( \delta'_\vartheta \) we find

\[ J_\vartheta + \phi_2(-a) + a \ln |a| + a = (n + \frac{1}{2}) \pi, \quad n = 0, 1, \ldots. \]

For Dirichlet boundary condition we therefore have the set of eigenvalue equations

\[ J_\varphi + \phi_2(a) - D(a) - a \ln |a| + a = (m + \frac{1}{2}) \pi, \quad m = 0, 1, \ldots, \]

\[ J_\varphi + \phi_2(-a) + a \ln |a| + a = (n + \frac{1}{2}) \pi, \quad n = 0, 1, \ldots. \]

(27)

The integrals defined by \( J_\vartheta \) and \( J_\varphi \) can be expressed in terms of elliptic integrals [6]

\[ J_\vartheta = \frac{-2 \mu}{\kappa \sqrt{(\nu + \frac{1}{2})^2 \kappa^2 - \mu}} \left\{ \Pi \left( \frac{\pi}{2}, \frac{(\nu + \frac{1}{2})^2 \kappa^2}{(\nu + \frac{1}{2})^2 \kappa^2 + \mu}, r_+ \right) \right\}, \]

and

\[ J_\varphi = \frac{-2 \mu}{\kappa' \sqrt{(\nu + \frac{1}{2})^2 \kappa^2 + \mu}} \left\{ \Pi \left( \frac{\pi}{2}, \frac{(\nu + \frac{1}{2})^2 \kappa^2 + \mu}{(\nu + \frac{1}{2})^2 \kappa^2}, r_+ \right) - K(r_+) \right\}, \]

for \( \mu > 0 \) and

\[ J_\vartheta = \frac{-2 \mu}{\kappa \sqrt{(\nu + \frac{1}{2})^2 \kappa^2 - \mu}} \left\{ \Pi \left( \frac{\pi}{2}, \frac{(\nu + \frac{1}{2})^2 \kappa^2}{(\nu + \frac{1}{2})^2 \kappa^2 + \mu}, r_- \right) - K(r_-) \right\}, \]

and

\[ J_\varphi = \frac{-2 \mu}{\kappa' \sqrt{(\nu + \frac{1}{2})^2 \kappa^2 + \mu}} \left\{ \Pi \left( \frac{\pi}{2}, \frac{(\nu + \frac{1}{2})^2 \kappa^2 + \mu}{(\nu + \frac{1}{2})^2 \kappa^2}, r_- \right) - K(r_-) \right\}, \]

for \( \mu < 0 \). In the above

\[ r_+ = \frac{\kappa}{\kappa'} \left( \frac{(\nu + \frac{1}{2})^2 \kappa^2 - \mu}{(\nu + \frac{1}{2})^2 \kappa^2 + \mu} \right)^{1/2}, \]

\[ r_- = \frac{\kappa'}{\kappa} \left( \frac{(\nu + \frac{1}{2})^2 \kappa^2 - \mu}{(\nu + \frac{1}{2})^2 \kappa^2 + \mu} \right)^{1/2}, \]

\( \Pi \) is the elliptic integral of the third kind, and \( K \) is the elliptic integral of the first kind. We also have the following identity for \( J_\vartheta \) and \( J_\varphi \):

\[ J_\vartheta + J_\varphi = (\nu + \frac{1}{2}) \pi. \]
B. Neumann boundary condition

For the Neumann boundary condition we have

\[ F_\varphi(0) = 0, \quad Q_\varphi(0) = 0, \]
\[ F_\varphi(\pi) = 0, \quad Q_\varphi(\pi) = 0. \]

The solution is thus odd with respect to \( \varphi = 0 \) and \( \varphi = \pi \)
where the \( \Theta \) solution is odd with respect to \( \vartheta = 0 \) and even
with respect to \( \vartheta = \pi \). A similar approach yields the eigenvalue equations:

\[ J_\varphi + \phi_2(a) + D(a) - a \ln|a| - a = (m + \frac{1}{2})\pi, \quad m = 0, 1, \ldots \]
\[ J_\varphi + \phi_2(-a) + D(a) - a = (n + \frac{1}{2})\pi, \quad n = 0, 1, \ldots \quad (29) \]

Note that Eqs. (27) and (29) are two parameter eigenvalue equations. This means that for a given value of \( n \) and \( m \) each set of equations must be solved simultaneously. We used the Newton-Raphson method in two dimensions to compute the eigenvalues, \( \nu \) and \( \mu \) for each boundary condition. The WKB eigenvalues agree remarkably well with the exact eigenvalues [1].

C. The WKB eigenfunctions

Once the eigenvalues are computed, the WKB eigenfunctions can be obtained from Eqs. (4) and (5). It is obvious that either of these solutions multiplied by a constant is still a solution. However, the amplitudes \( A_\vartheta \) and \( A_\varphi \) were determined in such a way that the WKB solutions can be compared with the exact solution and with the solution of the Weber equation without the need to multiply them by a constant. This proves convenient when comparing individual eigenfunctions. The complete normal mode solution of the wave equation for this problem involves a sum over the product of all eigenfunctions divided by the normalization constant that removes any amplitude ambiguities [3]. The WKB normalization constant is derived later in this paper.

For even solutions the amplitudes \( A_\vartheta \) and \( A_\varphi \) were determined in such a way that the WKB solutions can be compared with the exact solution and with the solution of the Weber equation without the need to multiply them by a constant. This proves convenient when comparing individual eigenfunctions. The complete normal mode solution of the wave equation for this problem involves a sum over the product of all eigenfunctions divided by the normalization constant that removes any amplitude ambiguities [3]. The WKB normalization constant is derived later in this paper.

For even solutions the amplitudes \( A_\vartheta \) and \( A_\varphi \) were determined in such a way that the WKB solutions can be compared with the exact solution and with the solution of the Weber equation without the need to multiply them by a constant. This proves convenient when comparing individual eigenfunctions. The complete normal mode solution of the wave equation for this problem involves a sum over the product of all eigenfunctions divided by the normalization constant that removes any amplitude ambiguities [3]. The WKB normalization constant is derived later in this paper.
(25) and (22) are determined such that \( z(0) = 1 \) and \( z'(0) = 0 \), where \( z \) can be either \( \Theta \) or \( \Phi \). For odd solutions, on the other hand, the initial conditions are \( z(0) = 0 \) and \( z'(0) = 1 \). Because the derivative of the solution is nonzero at the initial point, \( A_\Theta \) and \( A_\Phi \) must be multiplied by the scale factors \( d\Theta/dx \) and \( d\Phi/dx \), respectively. To see this, take, for example, \( \Theta(\theta) \), which near \( \theta = 0 \) reduces to the solution of the Weber equation, \( y(x) \). Then

\[
\frac{d\Theta}{d\theta} \bigg|_{\theta = 0} - \frac{dy(x)}{d\theta} \bigg|_{\theta = 0} = \frac{dy(x)}{dx} \bigg|_{x = 0} = 1.
\]

Since \( dy/dx \big|_{x = 0} \) is chosen to be unity, \( \Theta(\theta) \) multiplied by
\( d\theta/dx \) satisfies the above equation. The same is true for
\( \Phi(\varphi) \), \( d\theta/dx \) and \( d\varphi/dx \) may be obtained from Eq. (15).

As was pointed out before, we are particularly interested
in those solutions for which the turning points lie close to 0
or \( \pi \). In this case near 0 and \( \pi \) the solution can be
obtained from the power series solution of the Weber equation
given by Eqs. (18) and (19). For a given set of eigenvalues \( \{\nu, \mu\} \), \( a \)
determined from Eq. (10). Then the power series solution
are obtained by choosing \( \theta \) or \( \varphi \) as independent variables
and determining \( x \) from Eqs. (11) or (12), which is then used
in Eqs. (18) and (19). The results are shown in Fig. 2. By
only using a few terms, the power series solution overlays
the exact solution for \( \theta \) or \( \varphi \) close to zero and \( \pi \) and
smoothly connects with the WKB solution in each case. In
this way an excellent approximation to the exact solution can
be obtained for the entire interval between 0 and \( \pi \).

IV. SOLUTION FOR LARGE \( \nu \) AND SMALL \( \mu \)

It was pointed out earlier that the main contribution to the
scattering amplitude comes from those eigenfunctions that
correspond to small values of \( \mu \). It will be shown in this
section that when \( \mu/\nu^2 \ll 1 \), the expressions for the eigenval-
u
ues and the integrals appearing in the phases of the WKB
solutions can be approximated by simple algebraic functions.

First, consider the integral

\[
J_\theta = \int_0^\pi \left( \frac{(\nu + \frac{1}{2})^2 \kappa^2 \sin^2 \theta + \mu}{1 - \kappa^2 \cos \theta} \right)^{1/2} d\theta,
\]

\[
J_\varphi = \frac{2\kappa}{\kappa'} \left( \frac{\nu + \frac{1}{2}}{2} \right) \int_0^{\pi/2} \frac{\sqrt{\sin^2 \varphi + \alpha}}{\sqrt{1 + \xi \sin^2 \varphi}} d\varphi,
\]

where

\[
\alpha = \frac{\mu}{\kappa^2 (\nu + \frac{1}{2})^2} = \frac{2a\kappa'}{\kappa (\nu + \frac{1}{2})}, \quad \xi = \left( \frac{\kappa}{\kappa'} \right)^2.
\]

For small values of \( \alpha \) this integral is approximated by [7]

\[
J_\theta = (\nu + \frac{1}{2}) \arctan \frac{\kappa}{\kappa'} + a \ln[8\kappa \kappa' (\nu + \frac{1}{2})] - a \ln|a| + a + O(\alpha^2).
\]

Similarly,

\[
J_\varphi = (\nu + \frac{1}{2}) \arctan \frac{\kappa'}{\kappa} - a \ln[8\kappa \kappa' (\nu + \frac{1}{2})] + a \ln|a| - a + O(\alpha^2).
\]

By adding the eigenvalue equations for the Dirichlet boundary
condition, Eq. (27), and using Eq. (28) we find

\[
\nu = m + n + \frac{1}{4} + \frac{D(a)}{\pi}.
\]

For Neumann boundary condition we similarly find

\[
\nu = m + n + \frac{3}{4} - \frac{D(a)}{\pi}.
\]

By subtracting the eigenvalue equations for the Dirichlet
boundary condition and using Eqs. (30), (31), and (32) we find

\[
2 \left( m + n + \frac{3}{4} + \frac{D(a)}{\pi} \right) \left( \frac{\pi}{2} - \beta \right) + 2a \ln \left[ 8\kappa \kappa' \left( m + n + \frac{3}{4} + \frac{D(a)}{\pi} \right) \right] - 2\phi_2(a) + D(a) = \left( n - m + \frac{1}{4} \right) \pi.
\]

In the same way we find for the Neumann boundary condition

\[
2 \left( m + n + \frac{5}{4} - \frac{D(a)}{\pi} \right) \left( \frac{\pi}{2} - \beta \right) + 2a \ln \left[ 8\kappa \kappa' \left( m + n + \frac{5}{4} - \frac{D(a)}{\pi} \right) \right] - 2\phi_2(a) - D(a) = \left( n - m - \frac{1}{4} \right) \pi.
\]

In the above equations use has been made of the relations-

\[
\arctan \frac{\kappa'}{\kappa} = \frac{\beta}{2}, \quad \arctan \frac{\kappa}{\kappa'} = \frac{\pi}{2} - \frac{\beta}{2}.
\]

Equations (34) and (35) have the advantage that they only
depend on the parameter \( a \). This allows one to solve these
equations for \( a \) by performing a search in one dimension (as
opposed to two dimensions when the equations depend on \( \nu \)
as well) and then use Eqs. (32) and (33) to determine \( \nu \). The
eigenvalues obtained in this way are still in excellent agree-
ment with the exact eigenvalues. The phase of the WKB
solution for the \( \Theta \) equation, Eq. (4) is
In the above for \( dq \)

Using the WKB eigenvalue equations and the expressions

Similarly the phase of the boundary conditions

For the Dirichlet boundary condition we find

This can be written as

where

\[
\Psi(\varphi) = g(\varphi) - u(\varphi).
\]

For the Dirichlet boundary condition we find

and for the Neumann boundary condition we find

In the above \( g(\varphi) = f(\vartheta) \) with \( \kappa \) replaced by \( \kappa' \), \( \mu \) by \( -\mu \), and \( \delta_\vartheta \) by \( \delta_\varphi \). For small \( \omega \) it may be shown that

where \( a = \mu / 2 \kappa' (\nu + 1 / 2) \). A similar solution can be obtained for \( u(\varphi) \) by replacing \( \kappa \) with \( \kappa' \) and \( a \) with \( -a \).

\[
\Psi(\varphi) = g\left(\frac{\pi}{2}\right) - u(\varphi).
\]

\[
g\left(\frac{\pi}{2}\right) = m \frac{\pi}{2},
\]

\[
g\left(\frac{\pi}{2}\right) = \frac{1}{2} \left( m + \frac{1}{2} \right) \pi - D(a).
\]

\[
N = \int_0^\pi \int_0^\pi |\Theta_{\text{Lame}}(\vartheta)\Phi_{\text{Lame}}(\varphi)|^2 \sigma \, d\vartheta \, d\varphi.
\]

where

\[
\sigma = \frac{\kappa' \sin^2 \varphi + \kappa^2 \sin^2 \varphi}{\sqrt{(1 - \kappa'^2 \cos^2 \varphi)(1 - \kappa^2 \cos^2 \varphi)}}.
\]

Let us define

\[
\lambda_{\text{Lame}}(\vartheta) = \Theta_{\text{Lame}}^2(\vartheta) \quad \text{and} \quad \lambda_{\text{Lame}}(\varphi) = \Phi_{\text{Lame}}^2(\varphi),
\]

then by submitting these quantities in Eq. (38), we find

\[
N = \int_0^\pi \int_0^\pi \lambda_{\text{Lame}}(\vartheta) \frac{\kappa' \sin^2 \varphi}{\sqrt{1 - \kappa'^2 \cos^2 \varphi}} \, d\vartheta \, d\varphi
\]

\[
\times \int_0^\pi \lambda_{\text{Lame}}(\varphi) \frac{1}{\sqrt{1 - \kappa^2 \cos^2 \varphi}} \, d\varphi
\]

\[
+ \int_0^\pi \lambda_{\text{Lame}}(\varphi) \frac{\kappa^2 \sin^2 \varphi}{\sqrt{1 - \kappa^2 \cos^2 \varphi}} \, d\varphi
\]

\[
\times \int_0^\pi \lambda_{\text{Lame}}(\vartheta) \frac{1}{\sqrt{1 - \kappa^2 \cos^2 \varphi}} \, d\vartheta.
\]

The above integrals are of the general form

\[
\int \lambda_{\text{Lame}}(\varphi) \nu(\varphi) \, d\varphi,
\]

where \( \nu \) can be either \( \vartheta \) or \( \varphi \) and \( \nu(\varphi) \) represents the other functions appearing in Eq. (39). Now let us write

\[
\int_0^\pi \lambda_{\text{Lame}}(\vartheta) \nu(\vartheta) \, d\vartheta
\]

as

\[
\int_0^\pi \lambda_{\text{Lame}}(\vartheta) \nu(\vartheta) \, d\vartheta = \int_0^\pi \left[ \lambda_{\text{Lame}}(\vartheta) - \lambda_{\text{Lame}}^{\text{WKB}}(\vartheta) \right] \nu(\vartheta) \, d\vartheta
\]

\[
+ \int_0^\pi \lambda_{\text{Lame}}^{\text{WKB}}(\vartheta) \nu(\vartheta) \, d\vartheta,
\]

where \( \lambda_{\text{Lame}}^{\text{WKB}}(\vartheta) \) is the WKB solution of the Lamé equation.

The first integral can be written as
\[
\int_0^\pi \left[ \chi_{\text{Lame}}(\vartheta) - \chi_{\text{WKB}}(\vartheta) \right] \nu(\vartheta) d\vartheta \\
= \int_0^\Delta \left[ \chi_{\text{WKB}}(\vartheta) - \chi_{\text{Lame}}(\vartheta) \right] \nu(\vartheta) d\vartheta \\
+ \int_{\Delta}^{\pi} \left[ \chi_{\text{Lame}}(\vartheta) - \chi_{\text{WKB}}(\vartheta) \right] \nu(\vartheta) d\vartheta \\
+ \int_{\pi-\Delta}^\pi \left[ \chi_{\text{WKB}}(\vartheta) - \chi_{\text{Lame}}(\vartheta) \right] \nu(\vartheta) d\vartheta.
\]

Since for \( \vartheta \) close to 0 or \( \pi \) the Lamé equation reduces to the Weber equation, \( \Delta \) is chosen to be small enough such that for \( 0 < \vartheta < \Delta \) and \( \pi - \Delta < \vartheta < \pi \), the solution of the Lamé equation can be represented by the solution of the Weber equation, denoted by \( \chi_{\text{Weber}} \), and the WKB solution of the Lamé equation can be represented by the WKB solution of the Weber equation, denoted by \( \chi_{\text{WKB}} \). In the region \( \Delta < \vartheta < \pi - \Delta \), both the solution of the Lamé equation and the solution of the Weber equation can be represented by their WKB solutions:

\[
\chi_{\text{Lame}}(\vartheta) = \chi_{\text{WKB}}(\vartheta), \quad \chi_{\text{Weber}}(\vartheta) = \chi_{\text{WKB}}(\vartheta).
\]

Based on this argument the middle integral in the above equation is zero and in the other integrals \( \chi_{\text{Lame}} \) can be replaced by \( \chi_{\text{Weber}} \). resulting in

\[
\int_0^\pi \left[ \chi_{\text{Lame}}(\vartheta) - \chi_{\text{WKB}}(\vartheta) \right] \nu(\vartheta) d\vartheta \\
= \int_0^\Delta \left[ \chi_{\text{Weber}}(\vartheta) - \chi_{\text{WKB}}(\vartheta) \right] \nu(\vartheta) d\vartheta \\
+ \int_{\Delta}^\pi \left[ \chi_{\text{WKB}}(\vartheta) - \chi_{\text{Weber}}(\vartheta) \right] \nu(\vartheta) d\vartheta.
\]

Substituting these in Eq. (40) we find

\[
\int_0^\pi \chi_{\text{Lame}}(\vartheta) \nu(\vartheta) d\vartheta \\
= \int_0^\Delta \chi_{\text{Weber}}(\vartheta) \nu(\vartheta) d\vartheta \\
+ \int_{\Delta}^\pi \chi_{\text{WKB}}(\vartheta) \nu(\vartheta) d\vartheta.
\]

In the second integral in the above equation let \( \vartheta' = \pi - \vartheta \), so

\[
\int_{\pi-\Delta}^\pi \chi_{\text{Weber}}(\vartheta) \nu(\vartheta) d\vartheta \\
= \int_0^{\pi-\Delta} \chi_{\text{Weber}}(\vartheta') \nu(\vartheta') d\vartheta' \\
+ \int_0^\Delta \left[ \chi_{\text{Weber}}(\vartheta) - \chi_{\text{WKB}}(\vartheta) \right] \nu(\vartheta) d\vartheta.
\]

Next, let us write

\[
\int_0^\Delta \left[ \chi_{\text{Weber}}(\vartheta) - \chi_{\text{WKB}}(\vartheta) \right] \nu(\vartheta) d\vartheta \\
= \int_0^\infty \left[ \chi_{\text{Weber}}(\vartheta) - \chi_{\text{WKB}}(\vartheta) \right] \nu(\vartheta) d\vartheta \\
- \int_0^\Delta \left[ \chi_{\text{Weber}}(\vartheta) - \chi_{\text{WKB}}(\vartheta) \right] \nu(\vartheta) d\vartheta.
\]

For \( \vartheta > \Delta \), \( \chi_{\text{Weber}}(\vartheta) = \chi_{\text{WKB}}(\vartheta) \) and thus the second integral in the above equation is zero and

\[
\int_0^\Delta \left[ \chi_{\text{Weber}}(\vartheta) - \chi_{\text{WKB}}(\vartheta) \right] \nu(\vartheta) d\vartheta \\
= \int_0^\infty \left[ \chi_{\text{Weber}}(\vartheta) - \chi_{\text{WKB}}(\vartheta) \right] \nu(\vartheta) d\vartheta.
\]

Thus

\[
\int_0^\Delta \chi_{\text{Lame}}(\vartheta) \nu(\vartheta) d\vartheta \\
= \int_0^\Delta \chi_{\text{Weber}}(\vartheta) \nu(\vartheta) d\vartheta \\
+ \int_0^\Delta \left[ \chi_{\text{WKB}}(\vartheta) - \chi_{\text{Weber}}(\vartheta) \right] \nu(\vartheta) d\vartheta.
\]

Note that in the first integral the functions in the square brackets are valid near \( \vartheta = 0 \) and in the second integral they are valid near \( \vartheta = \pi \). In the equation for the normalization we either have

\[
\nu(\vartheta) = \frac{\kappa^2 \sin^2 \vartheta}{\sqrt{1 - \kappa^2 \cos^2 \vartheta}}
\]

or

\[
\nu(\vartheta) = \frac{1}{\sqrt{1 - \kappa^2 \cos^2 \vartheta}}.
\]

In the first case we have

\[
\int_0^\pi \chi_{\text{Lame}}(\vartheta) \frac{\kappa^2 \sin^2 \vartheta}{\sqrt{1 - \kappa^2 \cos^2 \vartheta}} d\vartheta \\
= \int_0^\infty \left[ \chi_{\text{Weber}}(\vartheta) - \chi_{\text{WKB}}(\vartheta') \right] \frac{\kappa^2 \sin^2 \vartheta}{\sqrt{1 - \kappa^2 \cos^2 \vartheta}} d\vartheta' \\
+ \int_0^\pi \chi_{\text{Lame}}(\vartheta) \frac{\kappa^2 \sin^2 \vartheta}{\sqrt{1 - \kappa^2 \cos^2 \vartheta}} d\vartheta.
\]
There is negligible contribution from the first two integrals on the right hand side because for small values of \( \vartheta, \sin \vartheta \approx 0 \) and for large values of \( \vartheta, \chi_{\text{Weber}}(\vartheta) = \chi_{\text{WKB}}(\vartheta) \). Therefore, we find

\[
\int_0^\pi \chi_{\text{Lame}}(\vartheta) \frac{\kappa^2 \sin^2 \vartheta}{\sqrt{1 - \kappa^2 \cos^2 \vartheta}} \, d\vartheta = \int_0^\pi \chi_{\text{WKB}}(\vartheta) \frac{\kappa^2 \sin^2 \vartheta}{\sqrt{1 - \kappa^2 \cos^2 \vartheta}} \, d\vartheta.
\]

In the second case

\[
\int_0^\pi \chi_{\text{Lame}}(\vartheta) \frac{1}{\sqrt{1 - \kappa^2 \cos^2 \vartheta}} \, d\vartheta = \int_0^\pi \left[ \chi_{\text{Weber}}(\vartheta) - \chi_{\text{WKB}}(\vartheta) \right] \frac{1}{\sqrt{1 - \kappa^2 \cos^2 \vartheta}} \, d\vartheta + \int_0^\pi \chi_{\text{WKB}}(\vartheta) \frac{1}{\sqrt{1 - \kappa^2 \cos^2 \vartheta}} \, d\vartheta.
\]

Here again there is negligible contribution from the first two integrals for large values of \( \vartheta \) since \( \chi_{\text{Weber}}(\vartheta) = \chi_{\text{WKB}}(\vartheta) \). For small values of \( \vartheta \)

\[
\frac{1}{\sqrt{1 - \kappa^2 \cos^2 \vartheta}} \approx \frac{1}{\kappa'},
\]

and thus we have

\[
\int_0^\pi \chi_{\text{Lame}}(\vartheta) \frac{1}{\sqrt{1 - \kappa^2 \cos^2 \vartheta}} \, d\vartheta = \frac{1}{\kappa'} \int_0^\infty \left[ \chi_{\text{Weber}}(\vartheta') - \chi_{\text{WKB}}(\vartheta') \right] d\vartheta' + \frac{1}{\kappa'} \int_0^\infty \left[ \chi_{\text{Weber}}(\vartheta') - \chi_{\text{WKB}}(\vartheta') \right] d\vartheta' + \int_0^\pi \chi_{\text{WKB}}(\vartheta) \frac{1}{\sqrt{1 - \kappa^2 \cos^2 \vartheta}} \, d\vartheta.
\]

Let us define

\[
\epsilon_{\theta} = \int_0^\infty \left[ \chi_{\text{Weber}}(\vartheta') - \chi_{\text{WKB}}(\vartheta') \right] d\vartheta',
\]

and

\[
\epsilon'_{\theta} = \int_0^\infty \left[ \chi_{\text{Weber}}(\vartheta') - \chi_{\text{WKB}}(\vartheta') \right] d\vartheta',
\]

then

\[
\int_0^\pi \chi_{\text{Lame}}(\vartheta) \frac{1}{\sqrt{1 - \kappa^2 \cos^2 \vartheta}} \, d\vartheta = \frac{\epsilon_{\theta}'}{\kappa'} + \int_0^\pi \chi_{\text{WKB}}(\vartheta) \frac{1}{\sqrt{1 - \kappa^2 \cos^2 \vartheta}} \, d\vartheta.
\]

Following a similar procedure using integrals involving the variable \( \varphi \) and substituting in Eq. (38) yields

\[
N = \int_0^\pi \frac{\kappa^2 \sin^2(\vartheta) \chi_{\text{WKB}}(\vartheta)}{\sqrt{1 - \kappa^2 \cos^2 \vartheta}} \, d\vartheta \times \left[ \frac{\epsilon_{\varphi} + \epsilon'_{\varphi}}{\kappa} + \int_0^\infty \frac{\chi_{\text{WKB}}(\varphi)}{\sqrt{1 - \kappa^2 \cos^2 \varphi}} \, d\varphi \right] \times \left[ \frac{\epsilon_{\varphi} + \epsilon'_{\varphi}}{\kappa'} + \int_0^\infty \frac{\chi_{\text{WKB}}(\varphi)}{\sqrt{1 - \kappa^2 \cos^2 \varphi}} \, d\varphi \right],
\]

where

\[
\epsilon_{\varphi} = \int_0^\infty \left[ \chi_{\text{Weber}}(\varphi) - \chi_{\text{WKB}}(\varphi) \right] d\varphi,
\]

and

\[
\epsilon'_{\varphi} = \int_0^\infty \left[ \chi_{\text{Weber}}(\varphi') - \chi_{\text{WKB}}(\varphi') \right] d\varphi',
\]

and \( C_{\theta} \) and \( C_{\varphi} \) are proportionality constants given by

\[
\int \psi_{\text{Lame}}(\theta)^2 d\theta = C_{\theta} \int \psi_{\text{Weber}}(x)^2 dx,
\]

and

\[
\int \psi_{\text{Lame}}(\theta)^2 d\theta = C_{\varphi} \int \psi_{\text{Weber}}(x)^2 dx.
\]
\[
\int \psi_{\text{Lame}}(\phi)^2 d\phi = C_\phi \int \psi_{\text{Weber}}(x) dx.
\] (44)

Expressions for \(C_\theta\) and \(C_e\) are derived in the Appendix. The evaluation of the integrals appearing in the expressions for \(e\)'s is quite lengthy. In this paper we only report the results. The interested reader may refer to Ref. [7] for a detailed derivation. We find the WKB normalization coefficient

\[
N = \frac{A_\mu^2 A_\varphi^2 \pi}{2(\nu + \frac{1}{2})^2 \kappa} \left[ \frac{1}{2} \ln \left( 8 \kappa \kappa' \left( \nu + \frac{1}{2} \right) \right) \right] = -\frac{1}{2} \text{Re} \left\{ \psi \left( \frac{1}{2} + i a \right) \right\} \pm \frac{\arcsin \kappa}{e^{\pi a} + e^{-\pi a}},
\] (45)

where \(\psi\) is the digamma function defined by \(\psi(z) = d\Gamma(z)/dz\). In the above equation the plus sign is used for the Dirichlet boundary condition and the minus sign is used for the Neumann boundary condition. Table I shows a comparison between the exact and the WKB normalization coefficients for a 90° PAS subject to Dirichlet boundary condition. In Table I

\[
\alpha_{\text{exact}} = \frac{\mu_{\text{exact}}}{(\nu_{\text{exact}} + \frac{1}{2})^2 \kappa^2}, \quad \alpha_{\text{WKB}} = \frac{\mu_{\text{WKB}}}{(\nu_{\text{WKB}} + \frac{1}{2})^2 \kappa^2},
\]

and \(n\) and \(m\) are defined in Eq. (27). As can be seen from Table I, for large values of \(n\), where the WKB solution is accurate, and small \(\alpha\), where the approximation leading to Eq. (45) is valid, \(N_{\text{WKB}}\) matches \(N_{\text{exact}}\) very closely.

**APPENDIX: THE DERIVATION OF \(C_\theta\) AND \(C_e\)**

We use the fact that for small angles the Lamé and the Weber equations have the same solution up to a multiplicative constant. First, note that the WKB solution of the Weber equation using the amplitude factors in Eqs. (20) and (21) in Eq. (17) is given by

\[
\psi_{\text{Weber}}(x) = \sqrt{2} \left( \frac{1 + \eta^2(-a)}{\eta(-a)} \right)^{1/2} \frac{1}{\sqrt{\chi^2 + 4a}} \times \cos \left( \int_0^x \frac{\sqrt{\chi^2 + 4a}}{2} d\chi \right).
\]

Then

\[
\int \psi_{\text{Weber}}(x)^2 dx = 2 \frac{1 + \eta^2(-a)}{\eta(-a)} \int \frac{1}{\sqrt{x^2 + 4a}} \\
\times \cos^2 \left( \frac{1}{2} \int_0^x \frac{\sqrt{x^2 + 4a}}{2} dx \right) dx.
\] (A1)

On the other hand we have

\[
\int \psi_{\text{Lame}}(\theta)^2 d\theta = A^2_\theta \int \cos^2 \left( \int_0^{\theta} \left( \frac{1 + \frac{1}{2} \kappa^2 \sin^2 \vartheta + \mu}{1 - \kappa^2 \cos^2 \vartheta} \right)^{1/2} d\vartheta \right) \frac{\cos \left( \int_0^{\theta} \frac{1}{\sqrt{\left( \nu + \frac{1}{2} \right)^2 \kappa^2 \sin^2 \vartheta + \mu}} d\vartheta \right)}{\sqrt{\left( \nu + \frac{1}{2} \right)^2 \kappa^2 \sin^2 \vartheta + \mu}} d\theta.
\]

Comparing Eqs. (A1) and (A2) with Eq. (43) yields

\[
C_\theta = \frac{A^2_\theta}{2 \kappa (\nu + \frac{1}{2})} \frac{\eta(-a)}{1 + \eta^2(-a)},
\]

where \(a = \mu/2\kappa \kappa' (\nu + \frac{1}{2})\). In a similar manner we find

\[
C_e = \frac{A^2_e}{2 \kappa' (\nu + \frac{1}{2})} \frac{\eta(a)}{1 + \eta^2(a)}.
\]