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APPLICATIONS OF GAUSSIAN BEAM TRACING TO TWO-AND THREE-DIMENSIONAL PROBLEMS IN OCEAN ACOUSTICS

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The Gaussian beam tracing method has recently received a great deal of attention for treating various problems in wave propagation. Principally, the technique competes with conventional ray tracing but has the advantages of not requiring eigenray computations and of being free of certain ray tracing artifacts such as infinitely high intensity levels at caustics and zero intensity levels in shadow zones. The method employs a fan of beams to approximate the field due to a point source in a medium with arbitrary variation of the wave speed(s). The central axis of the beam follows a path governed by the usual ray equations. These ray equations are augmented by additional differential equations which govern the evolution of the beam radius and curvature as a function of arclength. We consider two applications of Gaussian beam tracing. First, we treat the standard ocean acoustic problem of a "monochromatic" point source in a cylindrical waveguide with a sound speed, $c(r,z)$, which depends on both range and depth. Secondly, we consider the fully three-dimensional environment (with $c = c(x,y,z)$) employing beams which form a fan over both azimuthal and elevation-declination angles. Results for several oceanic scenarios are presented and compared to other solutions.

1. INTRODUCTION

Ray tracing methods play an important role in many areas of wave propagation. In principal the method is quite simple, however, in practice the development of a robust and efficient code is quite complicated. Amongst the difficulties one encounters are the problem of finding eigenrays, i.e. the rays which connect the source and receiver, singularities at caustics and shadow zones. Attempts have been made to circumvent these difficulties by making the rays diffuse. In particular, several ray codes associate a Gaussian intensity distribution with each ray with the width of the Gaussian specified by the user. This is the simplest form of Gaussian beam tracing.

In an attempt to provide a somewhat more formal basis for this technique Bucker[1] introduced a procedure in which the beam width and curvature evolved as a function of arclength along the ray. In essence these quantities evolved in a manner consistent with a Gaussian beam in a homogeneous medium.

More recently, a procedure has been introduced in the seismological literature in which the evolution of the beams is based on an asymptotic solution of the wave equation in the vicinity of a ray. This leads to a set of ordinary differential equations which are easily integrated along with the standard ray equations and yield the beam width and curvature as a function of arclength. This latter approach is nicely described in a paper by Červený, Popov and Pšenčík[2] who trace their ideas back to earlier papers by Babich, Kirpičnikova and Buldyrev. Similar procedures have also been used in quantum mechanics[3] and optics[4]. In the context of underwater acoustics problems we mention Ref. [5] which also includes a fairly complete set of references to the more recent literature.

In the following sections we will review the Gaussian beam tracing method for azimuthally symmetric ocean acoustic problems. This provides a simple introduction to the technique which is subsequently discussed for 3D problems. Finally, a simple deep-water sound speed profile is used to demonstrate the method for both 2D and 3D problems.

2. GAUSSIAN BEAMS IN TWO-DIMENSIONS

The steps of the Gaussian beam tracing method are basically 1) trace the central rays of the beams, 2) compute beam width, $L(s)$, and curvature, $K(s)$, describing the beam about each central ray, and 3) sum up the contributions of each beam to obtain the complex pressure at each receiver location of interest. Let us consider each of these steps in more detail.

The central rays of the beams satisfy the usual ray equations. Thus, introducing $(r(s), z(s))$ as the trajectory of a ray in cylindrical coordinates and $(\rho(s), \zeta(s))$ as the tangent vector to the ray one finds,

$$\frac{dr}{ds} = c\rho(s), \quad \frac{d\rho}{ds} = -\frac{1}{c^2}\frac{dc}{dr}, \tag{1}$$

$$\frac{dz}{ds} = c\zeta(s), \quad \frac{d\zeta}{ds} = -\frac{1}{c^2}\frac{dc}{dz}, \tag{2}$$

where s is arclength along the ray and $c(s)$ is the sound speed in the ocean which may vary both as a function of range and depth. This system of four ordinary differential equations may be integrated using any of the standard techniques, e.g. Runge-Kutta. Initial conditions are that the ray originates at the source location, (r_s, z_s) and with take-off angle α. That is,

$$r(0) = r_s, \quad \rho(0) = \frac{1}{c(0)}\cos\alpha, \tag{3}$$

$$z(0) = z_s, \quad \zeta(0) = \frac{1}{c(0)}\sin\alpha. \tag{4}$$

The next step is to construct a beam about each of the central rays. This is done by integrating a second set of ordinary differential equations:

$$\frac{dq}{ds} = c(s)p(s), \quad \frac{dp}{ds} = -\frac{c_{nn}}{c^2(s)}q(s), \tag{5}$$

where c_{nn} is the second derivative of the sound speed in a direction normal to the central ray. The beam solution is then given by

$$u^{beam}(s,n) = A\sqrt{\frac{c(s)}{rq(s)}}\exp[-i\omega(\tau(s) + 0.5\frac{p(s)}{q(s)}n^2)], \tag{6}$$

where A is an arbitrary constant, n is the normal distance from the central ray and

$$\tau(s) = \int_0^s \frac{1}{c(s')}ds' \tag{7}$$

is the phase delay along the ray. The source is 'monochromatic' with angular frequency ω. With complex initial conditions, p and q become complex and the real and imaginary parts of p/q may be related to beam curvature, K, and width, L via

$$L(s) = \sqrt{-2\omega\Im\{p/q\}},$$
$$K(s) = -c(s)\Re\{p(s)/q(s)\}.$$

The optimal choice of initial beam width and curvatu
is a matter of current research.

The final step is the summing up of all of the bea
to obtain the complex pressure. The weightings of t
beams are determined by considering a canonical pro
lem of a point source in a homogeneous medium. T
result is

$$u(r,z) = \sum_{l=1}^{N_\alpha} \delta\alpha \frac{1}{c_0}\exp(\frac{i\pi}{4})\sqrt{\frac{q_l(0)\omega\cos\alpha_l}{2\pi}}u_l^{beam}, \quad (\ldots$$

where $\delta\alpha$ is the angular spacing between beams a
u_l^{beam} denotes the beam with take-off angle α_l.

In order to evaluate this expression one needs to conv
a particular receiver point given in (r, z) coordina
into the ray-centered coordinates (s, n). This sligh
awkward procedure can be avoided by using a repres
tation of the beam directly in the Cartesian coordin
system as suggested by Madariaga[6]. The result is,

$$u^{beam}(r,z) = A\sqrt{\frac{c(r)}{rq(r)}}\exp[-i\omega\{\tau(r) + \frac{1}{c(r)}t_z\Delta\ldots$$
$$+\frac{1}{2}(\Delta z)^2(0.5\frac{p(r)}{q(r)}t_r^2 + 2\frac{c_n}{c^2}t_zt_r - \frac{c_r}{c^2}t_z^2)\}], \quad (\ldots$$

where $(t_r, t_z) = c(\rho, \zeta)$ is the local tangent vector to t
central ray. In addition, $\Delta z = (z_r - z(s))$ the distan
in the z-direction between the central ray of the bea
and the receiver. Note that if the beam is travelli
horizontally, so that $t_z = 0$ then the normal distan
is the same as Δz and we recover the original result
Eq. (6).

3. GAUSSIAN BEAMS IN THREE-DIMENSIONS

The extension to three dimensions is described by Bab
and Popov[7] and requires fairly minor modifications
the 2D algorithm. Let us again go through the th
steps required for constructing the beam solution.
usual we begin by tracing a set of rays however in t
3D case the rays form a fan over both azimuthal a
elevation-declination angles. The ray equations in
are given by

$$\frac{dx}{ds} = c\xi(s), \quad \frac{d\xi}{ds} = -\frac{1}{c^2}\frac{dc}{dx}, \quad (\ldots$$

$$\frac{dy}{ds} = c\eta(s), \quad \frac{d\eta}{ds} = -\frac{1}{c^2}\frac{dc}{dy}, \quad (\ldots$$

$$\frac{dz}{ds} = c\zeta(s), \quad \frac{d\zeta}{ds} = -\frac{1}{c^2}\frac{dc}{dz}, \quad (\ldots$$

with initial conditions,

$$x(0) = x_s, \quad \xi = \frac{1}{c(0)}\cos\alpha\cos\beta, \quad (15)$$

$$y(0) = y_s, \quad \eta = \frac{1}{c(0)}\cos\alpha\sin\beta, \quad (16)$$

$$z(0) = z_s, \quad \zeta = \frac{1}{c(0)}\sin\alpha. \quad (17)$$

Here (x_s, y_s, z_s) denotes the source coordinates and α and β denote the take-off angle in elevation-declination and azimuth respectively.

Once again, the construction of beams in the neighborhood of the central rays requires the integration of a system of auxilliary equations which in three-dimensions involve five components:

$$\frac{dQ}{ds} = \frac{1}{c^2}(-c_{mm}f + 2c_{mn}h - c_{nn}g), \quad (18)$$

$$\frac{dP}{ds} = -c(f+g), \quad (19)$$

$$\frac{df}{ds} = cP - \frac{1}{c^2}c_{nn}Q, \quad (20)$$

$$\frac{dg}{ds} = cP - \frac{1}{c^2}c_{mm}Q, \quad (21)$$

$$\frac{dh}{ds} = -\frac{1}{c^2}c_{mn}Q. \quad (22)$$

Here c_{mm} and c_{nn} are second derivatives of the sound speed in the two normal directions $\tilde{e}_m$ and $\tilde{e}_n$ where

$$\tilde{e}_m = (L^{-1}[c\xi\zeta\cos\phi + \eta\sin\phi], L^{-1}[c\eta\zeta\cos\phi - \xi\sin\phi], cL\cos\phi),$$

$$\tilde{e}_n = (L^{-1}[c\xi\zeta\sin\phi - \eta\cos\phi], L^{-1}[c\eta\zeta\sin\phi + \xi\cos\phi], -cL\cos\phi),$$

and $L = \sqrt{\xi^2+\eta^2}$. (These formulas are derived in Cerveny and Hron[8]). This ray-centered coodinate system $(\tilde{t}, \tilde{e}_m, \tilde{e}_n)$ is a rotating trihedral with rotation angle ϕ satisfying the differential equation:

$$\frac{d\phi}{ds} = \frac{1}{c(s)}\frac{\zeta(\eta c_x - \xi c_y)}{\xi^2+\eta^2} \quad (23)$$

which must also be integrated along the central ray of the beam.

Once this system is integrated one obtains a representation of the beam as,

$$u^{beam}(s,m,n) = A\sqrt{Q(s)}\exp\{-i\omega[\tau(s) + \frac{f(s)n^2 + 2h(s)mn + g(s)m^2}{2Q(s)}]\}. \quad (24)$$

This representation follows from application of the reduced delta matrix method to the equations given by Babich and Popov. Initial conditions are given by

$$(P, Q, f, g, h) = (1, \epsilon_1\epsilon_2, \epsilon_2, \epsilon_1, 0) \quad (25)$$

where ϵ_1 and ϵ_2 are the (complex) beam constants which characterize the beam width and curvature in directions $\tilde{e}_m$ and $\tilde{e}_n$ respectively.

Finally, the complex pressure is computed by summing up all of the individual beams with appropriate weighting,

$$u(r,z) = \sum_{k=1}^{N_\beta}\sum_{l=1}^{N_\alpha}\delta\beta\delta\alpha\frac{\sqrt{\epsilon_1\epsilon_2}}{2\pi\sqrt{c_0}^3}u_{kl}^{beam} \quad (26)$$

4. APPLICATIONS

We consider first an azimuthally symmetric environment for which the treatment in cylindrical coordinates is appropriate. The sound speed profile is given by

$$c(r,z) = 1500.0\{1.0 + \epsilon(r)[\tilde{z} - 1 + e^{-\tilde{z}}]\}, \quad (27)$$

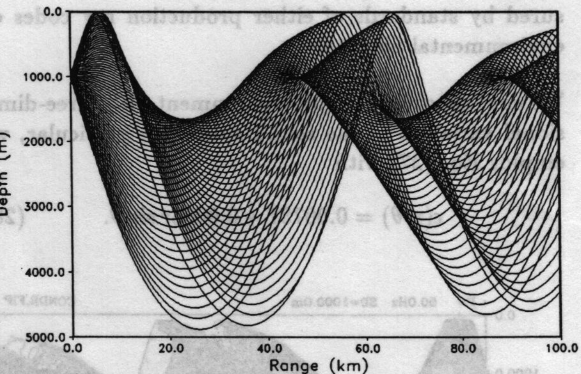

Figure 1: Ray trace for the deep-water sound speed profile.

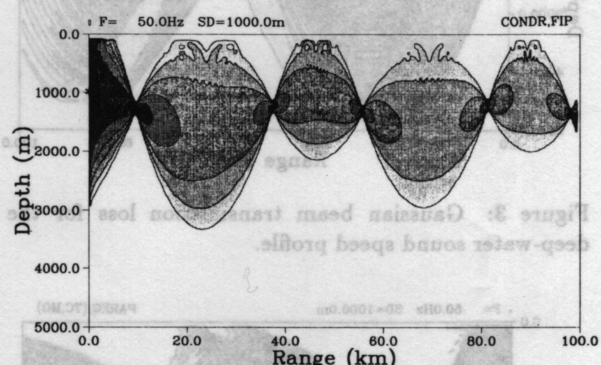

Figure 2: Intensity for the beam associated with the horizontally launched ray.

where $\epsilon = 0.00737 + 0.0003r$ and $\tilde{z} = 2(z - 1300)/1300$ is a scaled depth. This is a canonical deep water profile[5] modified in a somewhat contrived fashion to be range-dependent. A source depth of 1000 m and frequency of 50 Hz is selected. The various steps involved in computing the acoustic field are illustrated by the sequence of figures 1 – 3. Figure 1 shows a fan of rays emanating from the source position and spectrally windowed to exclude bottom bounce ray paths. Each ray becomes the central ray of a Gaussian beam as illustrated in Figure 2 for the horizontally launched ray. Note the focussing and subsequent defocussing that occurs as the beam passes through successive caustics of the ray trace. Finally, the contributions of all of the beams are summed to yield the pressure field displayed in Figure 3. (The grey scale plots in this paper employ levels 5 dB apart with black representing transmission loss of less than 70 dB.)

As mentioned earlier the solution so calculated is free of certain problems which standard ray codes have such as singularities at caustics and "drop-outs" where the computed intensity vanishes. Agreement with the parabolic equation solution (shown in Figure 4) is excellent measured by standards of either production ray codes or environmental uncertainty.

We next consider a simple environment with three-dimensional variation of the sound speed. In particular, we employ Eq. (27) with

$$\epsilon(r,\theta) = 0.00737 + 0.0003r\sin\theta. \qquad (28)$$

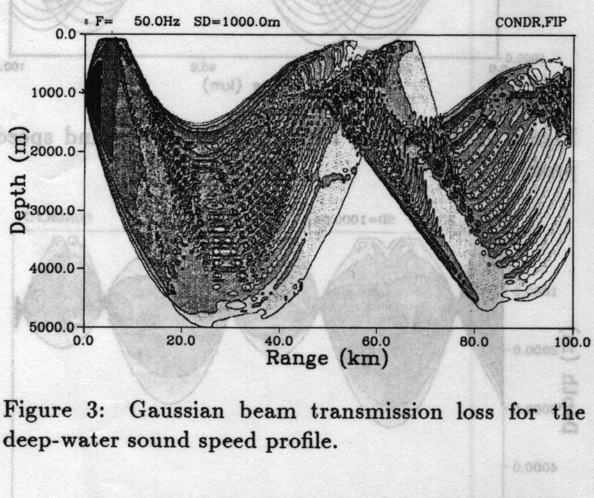

Figure 3: Gaussian beam transmission loss for the deep-water sound speed profile.

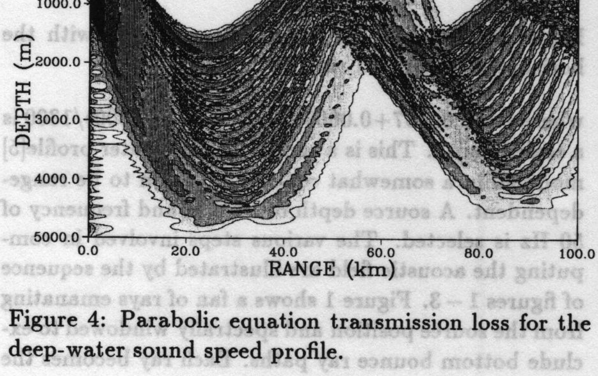

Figure 4: Parabolic equation transmission loss for the deep-water sound speed profile.

Along the radial $\theta = \pi/2$ we obtain the sound sp
used previously in the azimuthally symmetric envi
ment. The variation of ϵ causes a sinusoidal distor
of the convergence zone pattern which is visible in
transmission loss plot in Figure 5. The receiver de
in this case is 800 m so that the plot is a 'plan v
rather than the usual side view of transmission loss

Computation of full three-dimensional fields can req
considerable CPU time. In this case approximately
hours of CPU time were required on a VAX 8600
short-cut which can save significant time is to neg
horizontal refraction and simply apply the 2D code
dependently for each radial slice of the 3D problem
In fact, this '$N \times 2D$' option is also implemented
the 3D version of the code and for this particular
there is no significant difference between the 3D
$N \times 2D$ results. (CPU time however is reduced to
minutes for the $N \times 2D$ computation.) In genera
appears unlikely that ocean sound speed profile va
tion is significant enough to require the full 3D s
tion. On the other hand, bathymetric variation
may induce horizontal refraction and there is some
perimental evidence that this effect is important. S
three-dimensional models will provide an approach
assessing the significance of this effect in realistic oc
environments.

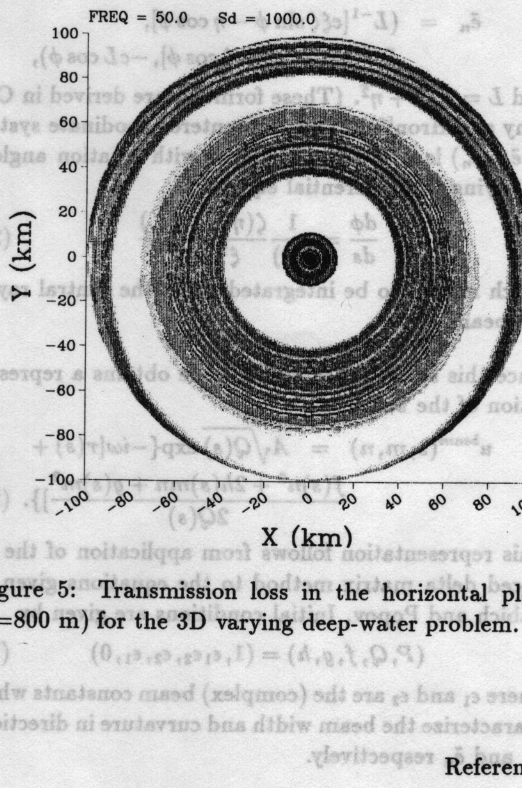

Figure 5: Transmission loss in the horizontal pl
(z=800 m) for the 3D varying deep-water problem.

Referen

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