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# Finite-Element Ray Tracing

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**Abstract**

The interesting acoustic modeling problems often push the practical limits of full-wave models. For instance, in acoustic tomography one needs to be able to predict the propagation of an acoustic pulse for successive realizations of 3D environments. For these types of problems ray methods continue to be attractive because of their speed. Unfortunately, existing codes are prone to a number of implementation difficulties which often degrade their accuracy. As a result most ray models are actually incapable of producing the ray theoretic result. We discuss a new method for implementing ray theory that uses a finite-element algorithm. This method is free of numerical artifacts affecting standard ray models and provides excellent agreement with more computationally intensive full-wave models.

# 1 Introduction

Ray tracing is one of the oldest methods for modeling sound propagation in the ocean. Newer so-called full-wave methods have supplanted ray tracing in many areas, however, ray models are still widely used. Their principle problem is a reputation for low-accuracy. Their principle strength is for high-frequency problems— full-wave models are often intolerably slow for such problems. Furthermore, the accuracy of ray models tends to improve at higher-frequencies.

Another key feature of ray models is their ability to handle broadband problems efficiently. In full wave models each new frequency usually requires a new model run. In contrast, for ray methods, many parts of the computation are independent of frequency, e.g., the ray paths and travel times. Furthermore, ray theory is directly applicable to range-dependent problems. Thus, a problem ideally suited to ray tracing might be a one with a high-frequency, broadband source in a range-dependent environment. This is precisely the scenario for two areas where ray models dominate: active sonar modeling and ocean acoustic tomography.

Gaussian beam tracing has been proposed as a natural replacement for ray tracing. In this approach beams are associated with the individual rays in a ray fan. The field at any given point is constructed by adding up the contributions of each of the beams. (For a more complete description see Refs. [1, 2].)

With properly chosen beam width and curvature, the Gaussian beam method is typically much more accurate. Furthermore, there is no need to locate eigenrays (rays that connect the source and receiver). Finding eigenrays is a nonlinear root-finding problem that tends to cause many problems in ray models. The key obstacle to the widespread acceptance of Gaussian beam tracing has been the difficulty in choosing the beam parameters.

In our work with the Gaussian beam method we often had difficulty obtaining quality ray solutions for comparison. (McGirr, et al.[3] reached a similar conclusion in their comparitive study of existing ray models for navy applications.) Most of the existing ray models introduce numerical artifacts that make the numerical results significantly worse than what we might term the ray-theoretical results. Thus, there is an intrinsic accuracy limitation in ray theory due to the high-frequency asymptotics. Beyond this, implementation difficulties were further degrading the results. Two common problems were a failure to incorporate a phase change at a caustic and failure to locate an eigenray.

In fairness, it should also be pointed out that these flaws in the ray models were often not critical because of the way the models were used (i.e., to produce illustrative ray paths). In addition, they were often used with simple reflection loss tables for the bottom loss. In the absence of detailed phase information about the bottom reflection coefficient, a caustic phase change is irrelevant for bottom reflected rays.

The ray tracing technique we describe here grew out of an attempt to develop an option in the Gaussian beam code that would allow us to produce a ray theoretic result. We found that through a simple modification of the beam algorithm this could be done in which 'hat-shaped' rather than Gaussian beams are constructed. The resulting technique ends up being analogous to the finite-element method and so we refer to the method as *finite-element ray* (FER) tracing.

The FER model inherits many of the appealing aspects of the Gaussian beam method. No eigenray tracing is required. This is particularly attractive for 3D problems where finding eigenrays is very complicated. Caustic phase changes are also easily included. Unlike Gaussian beam tracing the method recovers precisely the ray theoretic result. This is both good and bad. It is good in the sense that as pointed out earlier most ray models are actually incapable of producing the ray theoretic result. It is bad in the sense that standard ray theory is well-known to give singularities at caustics and foci. However, in general we find the results are surprisingly good.

# 2 The Algorithm

The basic technique is illustrated in Fig. 1. A fan of rays is traced from the source using the usual ray equations[4]:

$$\begin{aligned} \frac{dr}{ds} &= c\xi(s), & \frac{d\xi}{ds} &= -\frac{1}{c^2}\frac{dc}{dr}, \\ \frac{dz}{ds} &= c\zeta(s), & \frac{d\zeta}{ds} &= -\frac{1}{c^2}\frac{dc}{dz}, \end{aligned} \tag{1}$$

where $r(s), z(s)$ is the ray trajectory in cylindrical coordinates, $(\xi(s), \zeta(s))$ is a tangent to the ray and $s$ is arclength along the ray. The initial conditions specify the source location and the slope of the emitted ray:

$$\begin{aligned} r(0) &= r_s & \xi(0) &= \frac{\cos\alpha}{c(0)}, \\ z(0) &= z_s & \zeta(0) &= \frac{\sin\alpha}{c(0)}. \end{aligned} \tag{2}$$

where $(r_s, z_s)$ is the source coordinate and $\alpha$ is the ray take-off angle.

The intensity in ray codes is often calculated from the spreading of two adjacent rays. In our approach we reverse these steps. We use a differential

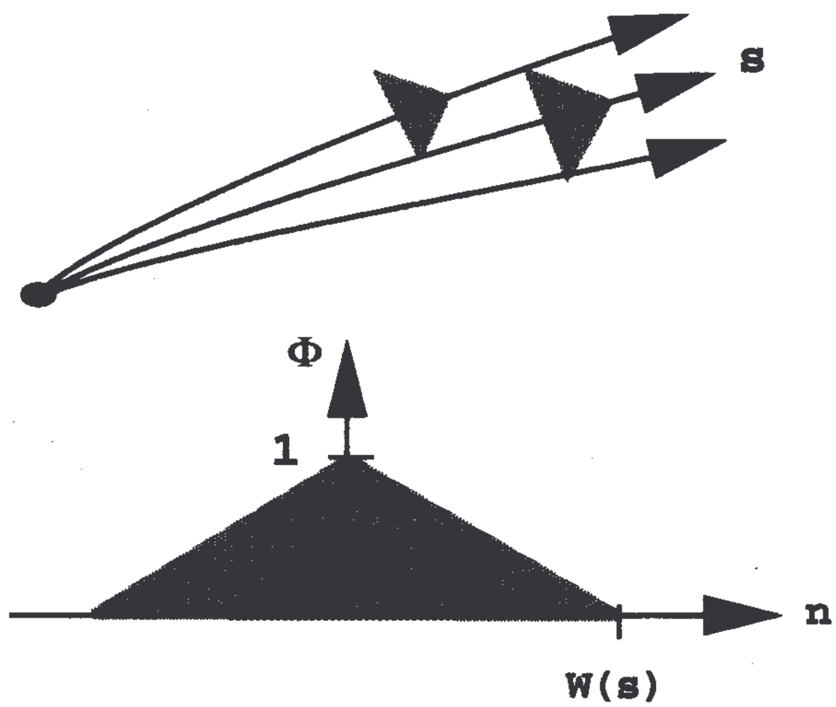


Figure 1: Construction of a finite-element ray.

equation that gives us directly the intensity along each ray. We then use that intensity to calculate a beam width so that the beam precisely fills the zone out to the neighboring ray.

The differential equations giving the intensity along a ray are the dynamic ray equations [5]:

$$\begin{aligned} \frac{dq}{ds} &= cp(s), \\ \frac{dp}{ds} &= -\frac{c_{nn}}{c^2(s)}q(s), \end{aligned} \tag{3}$$

where $c_{nn}$ is the derivative of the sound speed in a normal direction to the ray path.

The dynamic ray equations characterize the change in ray position due to general changes in the initial conditions for the ray. In our case we need the change in the area of the ray tube due to a change in ray take-off angle. The appropriate initial conditions for the dynamic ray equations are then:

$$q(0) = 0, \qquad p(0) = \frac{1}{c(0)}. \tag{4}$$

Then the amplitude along a ray is given by:

$$A(s) = \frac{1}{4\pi}\left|\frac{c(s)\cos\alpha}{r\,c(o)q(s)}\right|^{1/2}. \tag{5}$$

and the beam width $W(s)$ is

$$W(s) = \left|\frac{q(s)\,\delta\alpha}{c\xi}\right|, \tag{6}$$

where $\delta\alpha$ is the difference in angles between adjacent rays. Given the beam width we construct a hat-shaped function $\phi(n)$:

$$\phi(n) = \begin{cases} \frac{W(s)-n}{W(s)} & \text{for } n \leq W(s) \\ 0 & \text{else} \end{cases} \tag{7}$$

where $n(s)$ is the normal distance from the receiver to the central ray of the beam. The pressure field due to each beam is then given by:

$$P(s,n) = A(s)\phi(n)e^{i\omega\tau(s)}, \tag{8}$$

Here $\omega$ is the source frequency and $\tau(s)$ is the travel time along the ray:

$$\tau(s) = \tau(0) + \int_0^s \frac{1}{c(s')}ds'. \tag{9}$$

We select a fan of rays covering a domain of angles with a specified angular increment. For each take-off angle $\alpha$ we solve the ray equations (1) numerically to obtain the ray path. Simultaneously we integrate the dynamic ray equations to calculate the beam width $W(s)$ and amplitude $A(s)$. Thus we have constructed a beam about each central ray as defined by Eq. (8).

The field at any particular receiver point is obtained by summing up the contribution of each beam at that point. Thus the method is structurally almost identical to Gaussian beam tracing. However, the use of hat-shaped beams yields a ray-theoretical result. As such singularities will occur at caustics. These occur at points where $q(s)$ vanishes. We do not attempt to smooth out these singularities, however, we detect them by simply testing for a sign change in $q(s)$ and increment the phase by $\pi/2$.

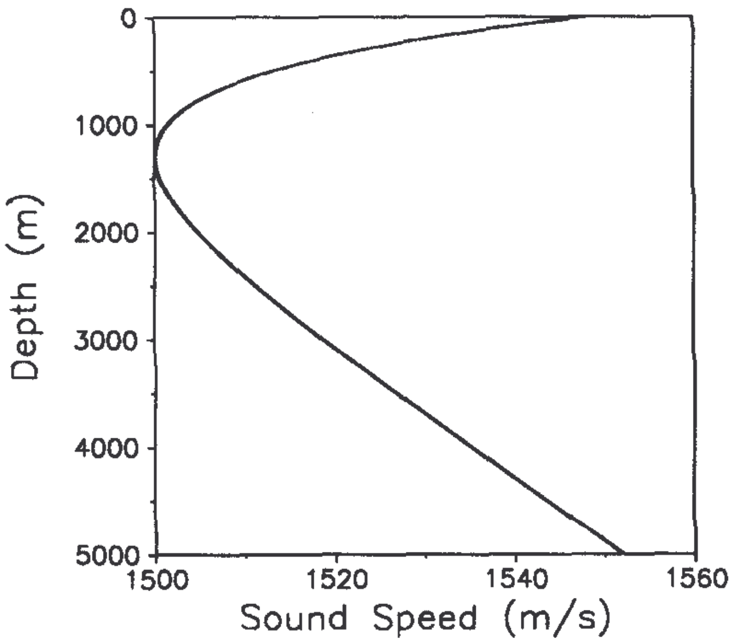


Figure 2: Sound speed plot for the Munk profile

# 3 Example

This scenario involves deep water and a canonical sound speed profile (the Munk profile[6]) that is often used as a test problem. The sound speed for this profile is given by:

$$c(z) = 1500.0[1.0 + \varepsilon(z' - 1 + e^{-z'})], \tag{10}$$

where

$$\varepsilon = 0.00737, \tag{11}$$

and the scaled depth $z\prime$ is given by

$$z\prime = \frac{2(z - 1300)}{1300}. \tag{12}$$

This sound-speed profile is plotted in Fig. 2. The corresponding ray trace for a source at 1000 m depth is shown in Fig. 3. To simplify the discussion we have restricted the ray fan to include only the waterborne ray paths.

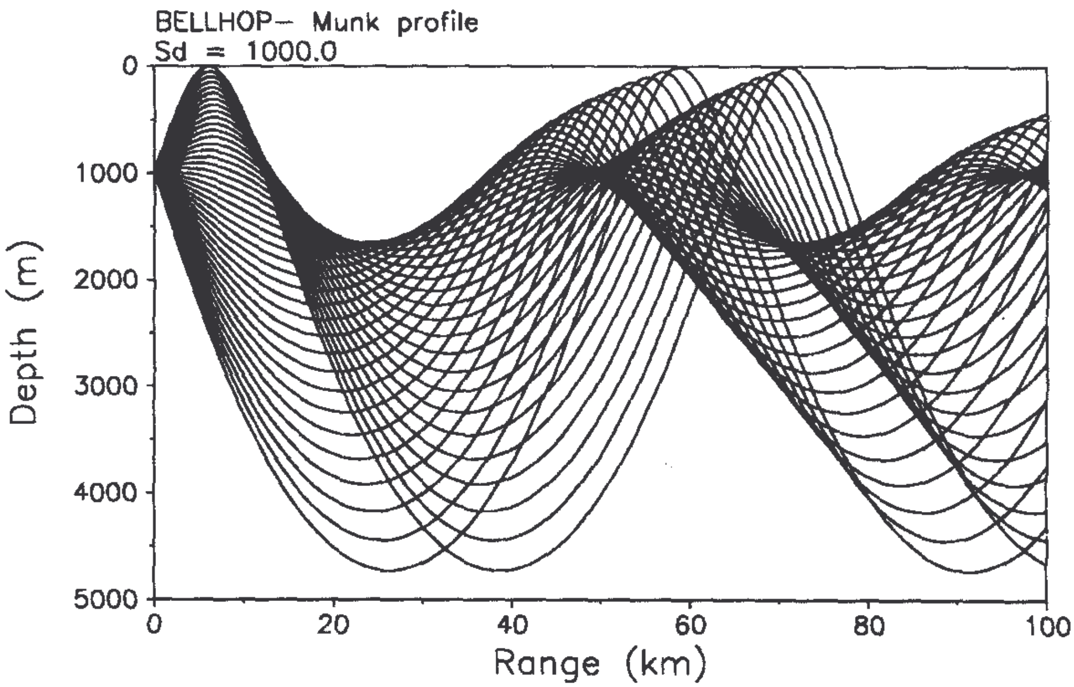


Figure 3: Ray trace for the Munk profile

The resulting transmission loss field is shown in Fig. 4 for a source frequency of 50 Hz. The upper plot is obtained using the FER model (BELLHOP). The lower plot is the reference solution obtained using a fast-field program (FFP)[7]. (The particular FFP used is SCOOTER which is described in Ref. [8].)

Most people would consider this an extremely low frequency problem for a ray model. Yet the agreement with the FFP results is excellent. Even fine details of the interference pattern are reproduced. The well-known flaws of the ray theoretic result are also visible. That is, we can see how the caustics of the ray trace emerge in the pressure field. However, these flaws are quite minor.

It is useful to look at a single slice to obtain a more quantitative measure of the agreement. In Fig. 5 we show a slice for a receiver depth of 800 m. The solid line is the reference solution obtain from the SCOOTER FFP and the dashed line is the FER result. Again we see excellent agreement apart from a few isolated zones where we pass near caustics of the ray field.

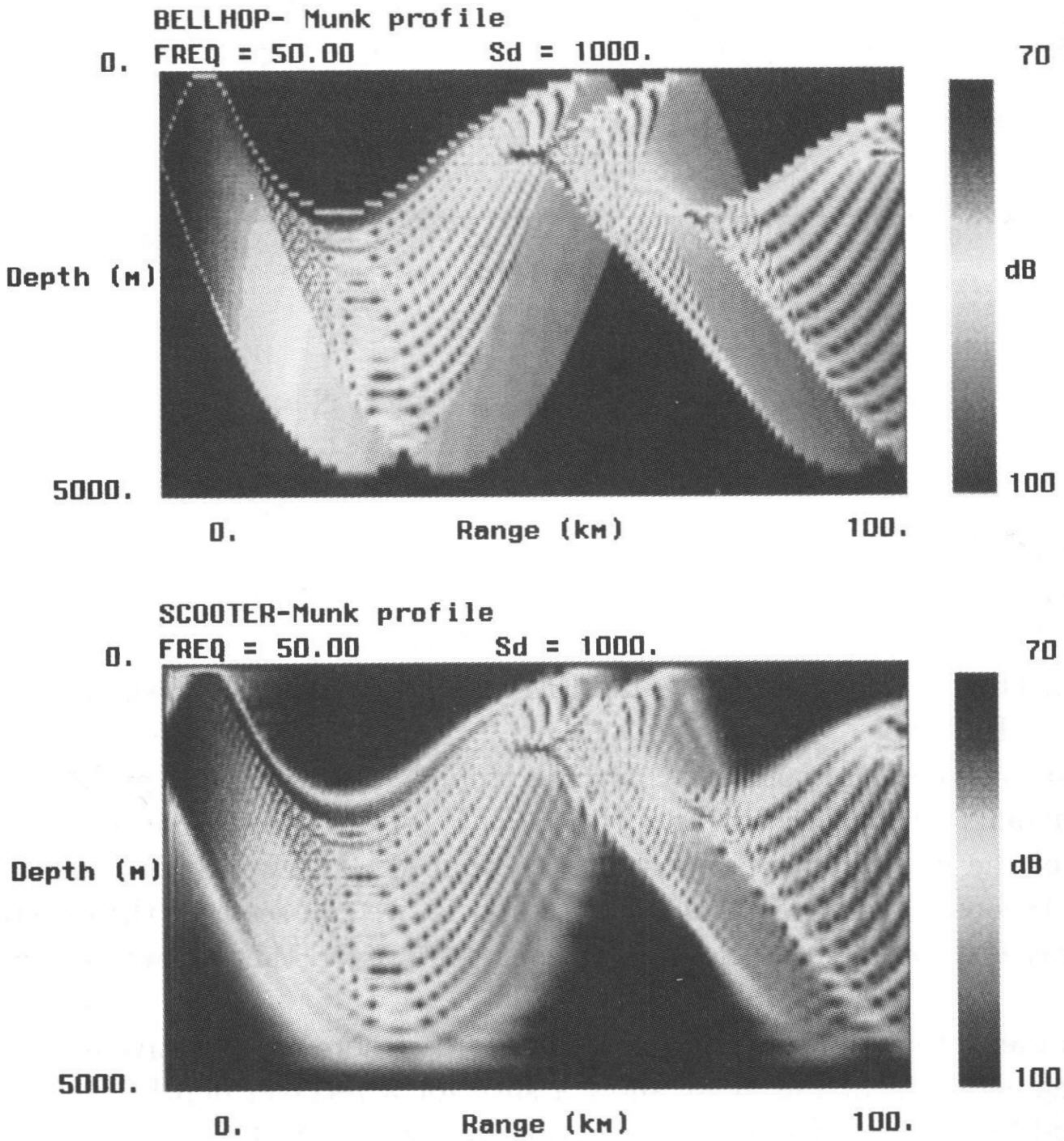


Figure 4: Transmission loss for the Munk profile using the FER method (upper) and an FFP program (lower).

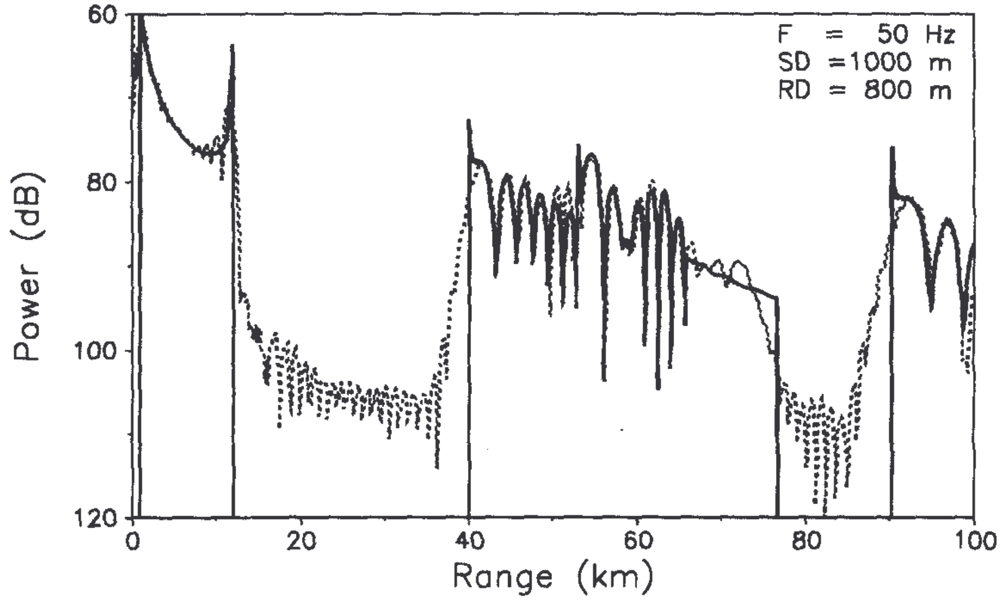


Figure 5: Transmission loss for the Munk profile for a receiver depth of 800 m using the FER method (dashed) and an FFP (solid).

# 4 Summary

We have applied the finite-element ray approach to a number of different scenarios varying from deep to shallow water and from Arctic to equatorial waters. The results presented here are in a sense representative. We experience no numerical difficulties with different cases, however the accuracy varies significantly depending on the density of caustics and the frequency. The accuracy varies in a systematic fashion with frequency in that the errors are confined to the vicinity of the caustics where the 'vicinity' is defined in terms of wavelengths.

The examples presented here are all range-independent, however, the code has been developed for the general range-dependent case. This is significant since many of the implementation difficulties are not present if a ray model is designed a priori to take advantage of range-independence.

In short, the FER approach provides a simple and efficient means of reconstructing the ray-theoretical result. Since it is implemented without caustic corrections it does retain the flaws inherent in the basic ray theory. On the other hand, most people working with ray models are accustomed to seeing ray theory together with implementation artifacts. With these removed we find that ray theory is surprisingly accurate.

# 5 Acknowledgment

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