A numerical method for bottom interacting ocean acoustic normal modes

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In this paper we present a finite difference method to numerically determine the normal modes for the sound propagation in a stratified ocean resting on a stratified elastic bottom. The compound matrix method is used for computing an impedance condition at the ocean–elastic bottom interface. The impedance condition is then incorporated as a boundary condition into the finite difference equations in the ocean, yielding an algebraic eigenvalue problem. For each fixed mesh size this eigenvalue problem is solved by a combination of efficient numerical methods. The Richardson mesh extrapolation procedure is then used to substantially increase the accuracy of the computation. Two applications are given to demonstrate the speed, accuracy, and efficiency of the method.

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INTRODUCTION

The amount of energy from a time periodic sound source that reaches the ocean's bottom depends on several parameters such as, the ocean's depth, and the source's frequency and location. Furthermore, the sound velocity profile in the ocean, which forms acoustic ducts, may trap much of the source's energy. If the interaction of the sound waves with the bottom is relatively unimportant the bottom is usually modeled as an acoustically rigid surface. However, for low-frequency sources and/or shallow oceans and/or weak acoustic ducts, the bottom interaction of the sound waves becomes a significant feature of the sound propagation. Then it is necessary to more carefully model the bottom material.

The material properties of the bottom may differ substantially depending on the geographical location. "Soft" bottoms have been modeled as fluid half-spaces neglecting the effect of shear waves,[1–3] or as fluid layers overlying uniform elastic half-spaces.[4–6] In other studies, the bottom is modeled by a sequence of uniform elastic layers with constant P and S wave velocities in each layer. The uniform layer models, which have been employed extensively in seismological studies, are known as the Thomson–Haskell, or propagator matrix method.[7,12] In this paper, we model the bottom as a *continuously stratified* elastic layer of finite thickness that is resting on a rigid half-space. This half-space corresponds to relatively rigid basement rocks.

The method of normal modes is a standard procedure that is used to solve sound propagation problems in stratified oceans. The resulting horizontal propagation numbers k and the normal modes are the eigenvalues and eigenfunctions, respectively, of a boundary value problem for an ordinary differential equation whose coefficients vary with the depth coordinate z, only. It is necessary to accurately determine these eigenvalues because errors in their values occur as phase shifts in the range dependence of the acoustic field. These errors can then rapidly degrade the accuracy of the normal mode representations as the distance from the source increases.

In Ref. 13 we have presented a finite difference method for numerically solving the eigenvalue problem for an acoustically rigid bottom. In this method standard and modified Richardson mesh extrapolation procedures are employed yielding a method which is extremely efficient over a wide range of accuracy requirements. In addition, a variety of well-known numerical procedures, such as Sturm sequences, the bisection method, and Newton's and Brent's methods, are employed to solve the resulting algebraic eigenvalue problems corresponding to each mesh width. In this paper, we present the necessary modifications in the numerical procedures so that it can be applied efficiently to solve the coupled problem of propagation in the ocean and in the elastic bottom. We employ the standard but not the modified Richardson extrapolation procedure of Ref. 13 in this paper, although it could be used to obtain increased accuracy. Other numerical procedures can be employed to solve the eigenvalue problem such as, shooting methods[14] and finite element methods using cubic splines.[15]

The formulation of the problem and the numerical procedures are described in Sec. I. The method is applied in Sec. II to two problems to demonstrate its accuracy and efficiency. The first problem corresponds to a shallow ocean with a relatively soft elastic bottom. In the second problem we consider a deep ocean where the Munk profile[16] is employed to model the sound speed stratification in the ocean. The P and S wave velocities in the bottom are assumed to be linear functions of the depth. The significance of the elastic bottom is demonstrated by a comparison in Sec. II with the results corresponding to a rigid ocean bottom.

Material absorption has not been included in our models of the ocean or the elastic bottom. The models can be modified to include this effect by introducing small imaginary components in the ocean and elastic wave speeds. Then

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the resulting problem could be solved numerically using methods similar to the present method.

I. FORMULATION

The differential equation for the acoustic normal pressure modes in a stratified ocean is given in

$$p'' + [\omega^2/c^2(z) - k^2]\, p = 0. \tag{1}$$

Primes denote differentiation with respect to the depth variable z, which is positively directed in the downward direction so that $z=0$, $z=D_1$, and $z=D_2$ correspond to the ocean surface, the ocean bottom, and the elastic layer bottom, respectively. Thus, $z=D_1$ is a fluid–elastic interface. In addition, ω is the circular frequency of the source, $c(z)$ is the sound speed, and k is the horizontal (range) wavenumber.

In the elastic bottom the corresponding elastic normal mode equations can be expressed as the following system of four first-order, ordinary differential equations[17]:

$$\mathbf{r}' = E\mathbf{r}. \tag{2}$$

Here $\mathbf{r}$ is the vector with components r_1, r_2, r_3, and r_4, which are defined by

$$ikr_1 \equiv u, \quad r_2 \equiv w, \quad ikr_3 \equiv \tau_{zx}, \quad r_4 \equiv \tau_{zz}, \tag{3a}$$

where, the quantities $u(z)$, $w(z)$, $\tau_{zx}(z)$, and $\tau_{zz}(z)$ are, with the factors $e^{i(kx-\omega t)}$ removed, the x displacement, the z displacement, the shear stress, and the normal stress, respectively. In addition, E is the 4×4 matrix defined by

$$E(z,k) = \begin{bmatrix} 0 & -1 & 1/(\rho c_s^2) & 0 \\ k^2\eta(z) & 0 & 0 & 1/(\rho c_p^2) \\ k^2\zeta(z) - \rho\omega^2 & 0 & 0 & -\eta(z) \\ 0 & -\omega^2\rho & 0 & 0 \end{bmatrix}, \tag{3b}$$

where the quantities $\eta(z)$ and $\zeta(z)$ in (3b) are defined by

$$\eta(z) \equiv \frac{c_p^2 - 2c_s^2}{c_p^2}, \quad \zeta(z) \equiv \frac{\rho[c_p^4 - (c_p^2 - 2c_s^2)^2]}{c_p^2}. \tag{3c}$$

Here, $c_p(z)$ and $c_s(z)$ are the P and S wave speeds in the elastic bottom and ρ is its density.

To complete the formulation of the eigenvalue problem we specify the following boundary and interfacial conditions:

$$p(0) = 0; \tag{4a}$$

$$\omega^2 r_2(D_1) = p'(D_1), \quad r_3(D_1) = 0, \quad r_4(D_1) = -p(D_1); \tag{4b}$$

$$r_1(D_2) = r_2(D_2) = 0. \tag{4c}$$

The condition (4a) implies that the ocean surface is free (pressure release); the conditions (4b) imply the continuity of the normal displacement and the two stresses at the interface $z=D_1$; finally, the conditions (4c) imply that the elastic layer, which is of thickness $D_2 - D_1$ is resting on a rigid basement at $z=D_2$. The eigenvalue problem is: for specified ω, ρ, $c(z)$, $c_p(z)$, and $c_s(z)$ determine the values of k for which (1)–(4) have nontrivial solutions.

To solve this problem numerically, we temporarily replace the elastic layer by the impedance condition at $z=D_1$,

$$g(k^2)p'(D_1) + f(k^2)\,p(D_1) = 0, \tag{5}$$

where the functions f and g are to be determined by the propagation in the bottom. The idea of using an impedance condition at the fluid–elastic interface has been previously employed in ocean acoustic studies (see e.g., Ref. 18 and references given therein). Then the bottom is usually modeled by uniform layers, thus permitting explicit representation of the waves in the bottom, resulting in simplifications in the analysis.

We first consider the ocean acoustic eigenvalue problem consisting of (1), (4a), and (5). We divide the interval $[0,D_1]$ into N_1 equals subintervals by the points $z_i = ih$, $i=0,1,\ldots,N_1$ where the mesh width $h = D_1/N_1$. Then using the standard three-point difference approximation to the second derivative in (1) and the centered difference approximation to the first derivative in the impedance condition (5) we obtain the algebraic eigenvalue problem

$$A(k^2)\mathbf{p} = 0 \tag{6}$$

as an approximation to the ocean acoustic eigenvalue problem. Here $\mathbf{p}$ is the N_1-dimensional vector with components $p_1, p_2, \ldots, p_{N_1}$. The $N_1 \times N_1$ tridiagonal matrix A is defined by

$$A \equiv \begin{bmatrix} a_1 - h^2k^2 & 1 & & & \\ 1 & a_2 - h^2k^2 & 1 & & \bigcirc \\ & \ddots & \ddots & \ddots & \\ \bigcirc & & 1 & a_{N_1-1} - h^2k^2 & 1 \\ & & & 2g(k^2) & g(k^2)(a_{N_1} - h^2k^2) - 2hf(k^2) \end{bmatrix}, \tag{7}$$

where the coefficients a_i are defined by

$$a_i \equiv -2 + h^2\omega^2/c^2(z_i), \quad i = 1,2,\ldots,N_1. \tag{8}$$

To determine the coefficients f and g in (5) we first obtain two linearly independent solutions $\mathbf{r}$ and $\mathbf{s}$ by integrating the modal equations (2) in the elastic layer from the rigid basement up to the interface with initial conditions given by

$$\mathbf{r}(D_2) = (0,0,1,0), \quad \mathbf{s}(D_2) = (0,0,0,1). \tag{9}$$

If k^2 is an eigenvalue then it is possible to form a linear combination, $C_1\mathbf{r} + C_2\mathbf{s}$, that satisfies the interface conditions at

$z = D_1$; that is

$$\begin{bmatrix} r_2 & s_2 \\ r_3 & s_3 \\ r_4 & s_4 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} p'/\omega^2 \\ 0 \\ -p \end{bmatrix}. \tag{10}$$

These are three linear algebraic equations for the three unknowns C_1, C_2, and k. The quantities r_j and $s_j, j = 2,3,4$, in (10) are components of **r** and **s**. By eliminating C_1 and C_2 from (10), we find that in order for the system to be solvable k must satisfy (5), where

$$f = \omega^2(r_3 s_2 - r_2 s_3), \quad g = r_3 s_4 - r_4 s_3. \tag{11}$$

Thus, the functions f and g can be computed by integrating (2) twice from the rigid bottom up to the interface with different initial conditions. However, because of exponential behavior in **r** and **s** as $z \to D_1$, these vectors are nearly linearly dependent (numerically) for z near D_1. This difficulty is resolved by using the compound matrix method.[12,19,20]

We define the variables $Y_1(z),...,Y_6(z)$ by

$$\begin{aligned} &Y_1 = r_1 s_2 - r_2 s_1, \quad Y_2 = r_3 s_4 - r_4 s_3, \\ &Y_3 = r_1 s_3 - r_3 s_1, \quad Y_4 = -(r_2 s_3 - r_3 s_2), \\ &Y_5 = -(r_1 s_4 - r_4 s_1), \quad Y_6 = -(r_2 s_4 - r_4 s_2). \end{aligned} \tag{12}$$

Thus, it follows from (11) and (12) that

$$f = \omega^2 Y_4, \quad g = Y_2, \quad \text{for } z = D_1. \tag{13}$$

By differentiating (12) and using (2) to eliminate the derivatives of **r** and **s** we obtain the following differential equation for the five-dimensional vector **Y**:

$$\mathbf{Y}' \equiv \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 0 & 1/(\rho c_s^2) & -1/(\rho c_p^2) \\ 0 & 0 & 0 & -\omega^2\rho & -[k^2 t(z) - \omega^2\rho] \\ 0 & 0 & 0 & 1 & (c_p^2 - 2c_s^2)/c_p^2 \\ k^2 t(z) - \omega^2\rho & 1/(\rho c_p^2) & -k^2(c_p^2 - 2c_s^2)/c_p^2 & 0 & 0 \\ \omega^2\rho & -1/(\rho c_s^2) & -2k^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{pmatrix}. \tag{14a}$$

Since the differential equation for Y_6 reduces to $Y_6 = -k^2 Y_2$ it has been eliminated from this system. The initial conditions at $z = D_2$ for **Y** are obtained by substituting (9) into (12) to give

$$Y(D_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{14b}$$

The initial value problem (14) is then integrated backwards in z using the modified midpoint method.[21,22] It is an explicit second-order integrator for first-order systems of the form $\mathbf{Y}' = \mathbf{f}(z,\mathbf{Y})$ and is given by

$$\begin{aligned} &\mathbf{y}_0 = \mathbf{y}(z_0), \quad \mathbf{y}_1 = \mathbf{y}_0 + h_b \mathbf{f}(z_0, \mathbf{y}_0), \\ &\mathbf{y}_{i+1} = \mathbf{y}_{i-1} + 2h_b \mathbf{f}(z_i, \mathbf{y}_i), \quad i = 1,2,...,N_2, \end{aligned} \tag{15}$$

where h_b is the mesh width for the bottom, and Y_{N_2} is obtained from

$$Y_{N_2} = (\mathbf{y}_{N_2-1} + 2\mathbf{y}_{N_2} + \mathbf{y}_{N_2+1})/4. \tag{16}$$

The final step (16) is a filter to remove the relatively weak instability in the integration. We employ the one-point filter (16) because we use only the values of the functions at this terminal point ($z_1 = D_1$) to solve the eigenvalue. The number of subintervals in the bottom N_2 must be even. We employ this method because it is explicit, it has second-order accuracy, and finally because of its speed.

To solve the eigenvalue problem (1)–(4) for fixed mesh widths h and h_b, we first solve (14) for a given k^2 and then use (13) to obtain f and g. Then we obtain the determinant of (7) by the following recursion

$$\begin{aligned} &p_0 = 0, \quad p_1 = 1, \\ &p_i = (a_i - h^2 k^2) p_{i-1} - p_{i-2}, \quad i = 2,...,N_1 - 1, \\ &d(k^2) = [\, g(k^2)(a_{N_1} - h^2 k^2) + 2hf(k^2)\,]\, p_{N_1-1} \\ &\qquad - 2g(k^2)\, p_{N_1-2}. \end{aligned} \tag{17}$$

This is equivalent to shooting down from the ocean surface to the interface and computing $f(k^2)\,p(D_1) + g(k^2)p'(D_1)$. Subsequently k^2 is adjusted such that $d(k^2)$ vanishes by using the secant method, as we now describe.

In Ref. 13, where the boundary condition at $z = D_1$ is $p'(D_1) = 0$, it was possible to employ the Sturm sequence method to obtain isolating intervals for the positive eigenvalues of the resulting algebraic system, thus generating an initial guess for the coarsest mesh. However, to the authors' knowledge, the Sturm sequence procedure has not been extended to the present algebraic eigenvalue problem (6). Thus, we have employed the following method to obtain the algebraic eigenvalues for the coarsest mesh.

We apply the secant method beginning at a $k = k_{\max}$ equal to the upper bound on the eigenvalues. A good esti-

TABLE I. (a) Numerical eigenvalues for the shallow water problem. (b) Errors.

(a) $K^2(p)=100 \times k^2(p)$

N1/N2	100/300	150/450	200/600	300/900	
ET=	7.9870	5.7940	4.7920	6.5560	
j	K²(1)	K²(2)	K²(3)	K²(4)	Exact
1	1.7805178446	1.7806674179	1.7807197867	1.7807571989	1.7807871324
2	1.6848824426	1.6854136651	1.6855997232	1.6857326642	1.6858390427
3	1.5287832832	1.5297586069	1.5301003437	1.5303445619	1.5305400092
4	1.3164975109	1.3177504076	1.3181895437	1.3185034122	1.3187546275
5	1.0562214624	1.0573243701	1.0577109025	1.0579871623	1.0582082698
6	0.7688670707	0.7692402970	0.7693704899	0.7694633464	0.7695375485
7	0.4831481449	0.4824747429	0.4822376204	0.4820677931	0.4819316589
8	0.3851153068	0.3851065403	0.3851034540	0.3851012435	0.3850994716
9	0.3582958781	0.3582725441	0.3582643672	0.3582585232	0.3582538459
10	0.3193730866	0.3193449329	0.3193351361	0.3193281560	0.3193225825
11	0.2884255951	0.2882465840	0.2881841900	0.2881397014	0.2881041566
12	0.2724371949	0.2721226385	0.2720129433	0.2719347156	0.2718722087
13	0.2465595600	0.2462779763	0.2461792850	0.2461087480	0.2460522925
14	0.1851786040	0.1847981253	0.1846653715	0.1845706765	0.1844949975
15	0.1381986509	0.1375707205	0.1373512051	0.1371944841	0.1370691507
16	0.0836556721	0.0819155609	0.0813270671	0.0809132325	0.0805860636
17	0.0486839034	0.0462087151	0.0453006868	0.0446384773	0.0441004226
18	0.0032876241	0.0024746046	0.0022004628	0.0020083102	0.0018569072

(b) $e(p)=k^2-k^2(p)$

N1/N2	100/300	150/450	200/600	300/900
ET=	7.9870	5.7940	4.7920	6.5560
j	e(1)	e(2)	e(3)	e(4)
1	2.7E-06	1.2E-06	6.7E-07	3.0E-07
2	9.6E-06	4.3E-06	2.4E-06	1.1E-06
3	1.8E-05	7.8E-06	4.4E-06	2.0E-06
4	2.3E-05	1.0E-05	5.7E-06	2.5E-06
5	2.0E-05	8.8E-06	5.0E-06	2.2E-06
6	6.7E-06	3.0E-06	1.7E-06	7.4E-07
7	-1.2E-05	-5.4E-06	-3.1E-06	-1.4E-06
8	-1.6E-07	-7.1E-08	-4.0E-08	-1.8E-08
9	-4.2E-07	-1.9E-07	-1.1E-07	-4.7E-08
10	-5.1E-07	-2.2E-07	-1.3E-07	-5.6E-08
11	-3.2E-06	-1.4E-06	-8.0E-07	-3.6E-07
12	-5.6E-06	-2.5E-06	-1.4E-06	-6.3E-07
13	-5.1E-06	-2.3E-06	-1.3E-06	-5.6E-07
14	-6.8E-06	-3.0E-06	-1.7E-06	-7.6E-07
15	-1.1E-05	-5.0E-06	-2.8E-06	-1.3E-06
16	-3.1E-05	-1.3E-05	-7.4E-06	-3.3E-06
17	-4.6E-05	-2.1E-05	-1.2E-05	-5.4E-06
18	-1.4E-05	-6.2E-06	-3.4E-06	-1.5E-06

mate for $k_{\max}$ is given by $\omega/\min(c, c_p, c_s)$. It can be shown that when the roots are real the secant method converges to the largest root. In addition, if the "shift" suggested by the secant method at each step is doubled then the secant method can at most cross over one root. Thus when the sign changes one can switch back to the standard secant method and still be assured of convergence to the desired root.[23] The next largest root is found by deflating all previous known roots. The deflation is accomplished by expressing the determinant with known roots divided out as follows:

$$d(k^2) = d_1(k^2)\left(\prod_j \frac{k^2}{k^2 - k_j^2}\right). \tag{18}$$

As in Ref. 13, we apply the Richardson mesh extrapolation method to obtain improved estimates of the eigenvalues of the continuous problem from the approximations of the eigenvalues of the algebraic problem. Thus, if the converged numerical value of the jth eigenvalue of the algebraic problem with mesh width h is denoted $k_j^2(h)$, then we have[24]

$$k_j^2(h) = (k_j^0)^2 + b_2 h^2 + b_4 h^4 + \cdots . \tag{19}$$

Here, $(k_j^0)^2$ is the Richardson approximation to the jth eigenvalue of the continuous problem. It is then determined from the algebraic system that results from applying (19) to a sequence of successively finer meshes $\{h_j\} = h_1, h_2, \ldots, h_m$. Since this approximation depends on the sequence of the mesh widths that is employed we denote the Richardson approximation corresponding to the meshes $h_p, h_{p+1}, \ldots, h_{p+q}$ by $(k_j^0)^2\,(p,\ldots,q)$.

We wish to emphasize that the deflation procedure is

TABLE II. (a) Richardson extrapolations for the shallow water problem. (b) Errors.

(a) $K^2(p,\ldots,q)=100 \times k^2(p,\ldots,q)$

N1/N2	100/300	150/450	200/600	300/900	
ET=	7.9870	5.7940	4.7920	6.5560	
j	K²(1)	K²(1,2)	K²(1,2,3)	K²(1,2,3,4)	Exact
1	1.7805178446	1.7807870766	1.7807871316	1.7807871324	1.7807871324
2	1.6848824426	1.6858386431	1.6858390400	1.6858390427	1.6858390427
3	1.5287832832	1.5305388658	1.5305400043	1.5305400092	1.5305400092
4	1.3164975109	1.3187527249	1.3187546214	1.3187546275	1.3187546275
5	1.0562214624	1.0582066962	1.0582082648	1.0582082698	1.0582082698
6	0.7688670707	0.7695388780	0.7695375483	0.7695375485	0.7695375485
7	0.4831481449	0.4819360213	0.4819316578	0.4819316589	0.4819316589
8	0.3851153068	0.3850995272	0.3850994719	0.3850994716	0.3850994716
9	0.3582958781	0.3582538770	0.3582538465	0.3582538459	0.3582538459
10	0.3193730866	0.3193224099	0.3193225836	0.3193225825	0.3193225825
11	0.2884255951	0.2881033751	0.2881041670	0.2881041566	0.2881041566
12	0.2724371949	0.2718709934	0.2718722109	0.2718722088	0.2718722087
13	0.2465595600	0.2460527093	0.2460522917	0.2460522925	0.2460522925
14	0.1851786040	0.1844937424	0.1844950031	0.1844949974	0.1844949975
15	0.1381986509	0.1370683762	0.1370691693	0.1370691506	0.1370691507
16	0.0836556721	0.0805234721	0.0805860855	0.0805860745	0.0805860636
17	0.0486839034	0.0442285645	0.0441014410	0.0441004263	0.0441004226
18	0.0032876241	0.0018241889	0.0018559300	0.0018568930	0.0018569072

(b) $e(p) = k^2-k^2(p,\ldots,q)$

N1/N2	100/300	150/450	200/600	300/900
ET=	7.9870	5.7940	4.7920	6.5560
j	e(1)	e(1,2)	e(1,2,3)	e(1,2,3,4)
1	2.7E-06	5.6E-10	7.5E-12	2.1E-14
2	9.6E-06	4.0E-09	2.7E-11	7.0E-14
3	1.8E-05	1.1E-08	4.9E-11	1.2E-13
4	2.3E-05	1.9E-08	6.1E-11	1.2E-13
5	2.0E-05	1.6E-08	4.9E-11	8.1E-14
6	6.7E-06	-1.3E-08	2.5E-12	1.2E-14
7	-1.2E-05	-4.4E-08	1.2E-11	4.8E-14
8	-1.6E-07	-5.6E-10	-3.4E-12	-2.5E-16
9	-4.2E-07	-3.1E-10	-5.9E-12	-5.7E-15
10	-5.1E-07	1.7E-09	-1.1E-11	3.2E-14
11	-3.2E-06	7.8E-09	-1.0E-10	3.0E-13
12	-5.6E-06	1.2E-08	-2.2E-11	-2.6E-13
13	-5.1E-06	-4.2E-09	8.1E-12	-1.3E-14
14	-6.8E-06	1.3E-08	-5.7E-11	9.2E-14
15	-1.1E-05	7.7E-09	-1.9E-10	1.0E-12
16	-3.1E-05	6.3E-07	-2.2E-10	-1.1E-10
17	-4.6E-05	-1.3E-06	-1.0E-08	-3.7E-11
18	-1.4E-05	3.3E-07	9.8E-09	1.4E-10

used to obtain approximations for the algebraic eigenvalues only for the coarsest mesh. Initial guesses for the eigenvalues for the second and subsequent meshes are obtained by using the Richardson extrapolation procedure, but extrapolating to the desired mesh size, as we did in Ref. 13.

Once the eigenvalues have been computed to the desired accuracy, the ocean eigenfunctions are found by inverse iteration.[23] In some applications it is desirable to obtain eigenfunctions $\mathbf{r}(z)$ within the elastic layer. These elastic eigenfunctions may then be obtained by incorporating difference equations for the elastic layer into the matrix and then applying inverse iteration to the entire matrix. For our computations of the elastic eigenfunctions we have employed the trapezoidal method to obtain these difference equations in the bottom. The computation of the elastic eigenfunctions using the compound matrix method presents certain difficulties that we wished to avoid.

II. APPLICATIONS OF THE METHOD

Two applications of our method are now presented to demonstrate its accuracy, speed, and versatility. In the first problem we consider a shallow ocean, with a constant sound speed of 1500 m/s, and constant S- and P-wave speeds of 700 m/s, and 1700 m/s, respectively, in the bottom, i.e., we consider an isovelocity ocean and bottom. The ocean depth is 300 m and has a density of 1 g/cm^3 while the bottom layer has a thickness of 200 m and a density of 2 g/cm^3. The circular frequency is $\omega = 30\ \pi$/s.

The numerically determined eigenvalues and corresponding errors are presented in Table I. The exact eigenvalues for this problem were determined by a compound matrix formulation of the Thomson–Haskell method. The error table reflects the second-order convergence in that doubling the number of mesh points reduces the error by a factor of about 4. The extrapolations and their errors are presented in Table II. In contrast to the unextrapolated eigenvalues, the

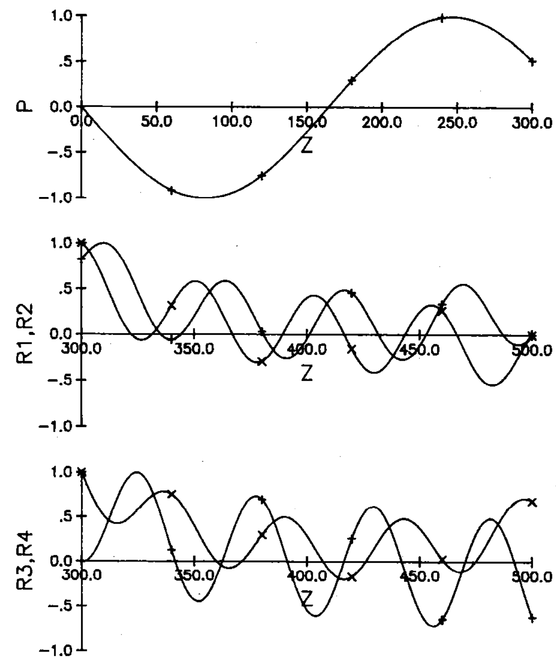

FIG. 2. Mode 9 for the shallow water problem. $R_1 = 1.57E+4\ r_1(+)$, $R_2 = -5.44E+5\ r_2(\times)$, $R_3 = -2.92E-1\ r_3(+)$, $R_4 = -1.93E-4\ r_4(\times)$.

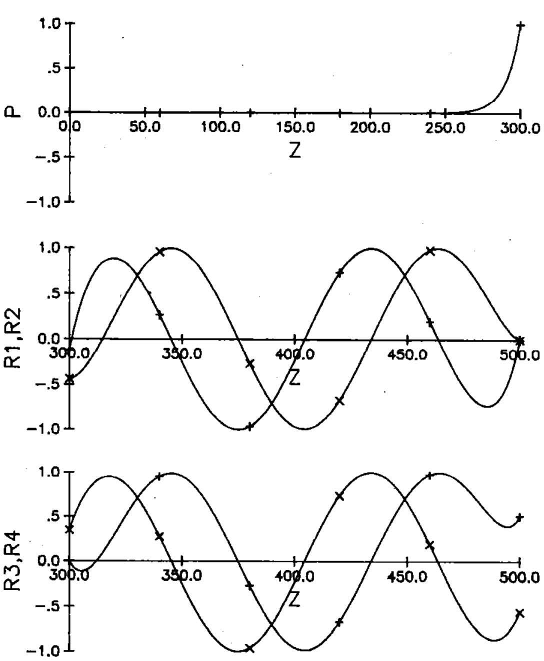

FIG. 1. Mode 3 for the shallow water problem. $R_1 = -1.04E+4\ r_1(+)$, $R_2 = -3.60E+4\ r_2(\times)$, $R_3 = -4.46E-2\ r_3(+)$, $R_4 = -3.47E-1\ r_4(\times)$.

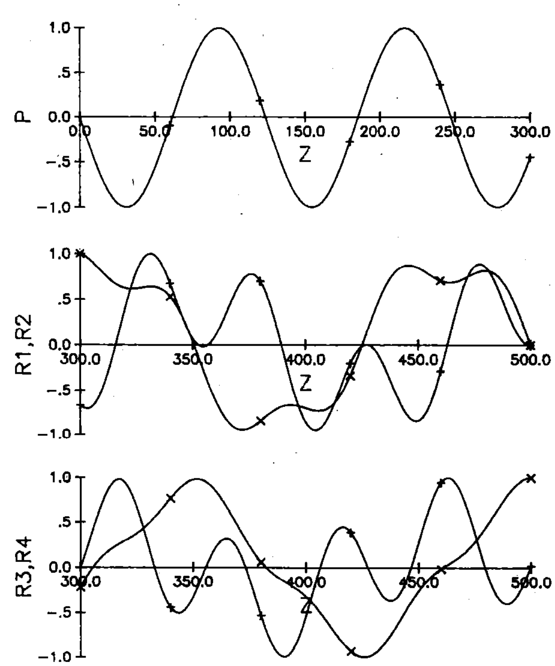

FIG. 3. Mode 15 for the shallow water problem. $R_1 = 4.97E+3\ r_1(+)$, $R_2 = 1.96E+5\ r_2(\times)$, $R_3 = 4.43E-2\ r_3(+)$, $R_4 = -4.93E-1\ r_4(\times)$.

TABLE III. (a) Numerical eigenvalues for the Munk profile with an elastic bottom. (b) Errors.

(a) $K^2(p)=10^4 \times k^2(p)$

N1/N2	200/200	300/300	400/400	600/600	
ET=	8.9040	7.7140	6.6490	8.3590	
j	$K^2(1)$	$K^2(2)$	$K^2(3)$	$K^2(4)$	Exact
1	2.9038346179	2.9043777763	2.9045717303	2.9047115148	2.9048240973
2	2.7853806272	2.7853796965	2.7853793709	2.7853791383	2.7853789522
3	2.7442802497	2.7442758805	2.7442743516	2.7442732597	2.7442723862
4	2.7058505588	2.7058401277	2.7058364780	2.7058338716	2.7058317867
5	2.6688852743	2.6688658371	2.6688590395	2.6688541857	2.6688503038
6	2.6298754553	2.6298386927	2.6298258400	2.6298166640	2.6298093260
7	2.5846045345	2.5845339025	2.5845092028	2.5844915671	2.5844774627
8	2.5314592539	2.5313318617	2.5312872953	2.5312554689	2.5312300119
9	2.4703582714	2.4701445959	2.4700698176	2.4700164073	2.4699736807
10	2.4014708645	2.4011337395	2.4010157242	2.4009314209	2.4008639743
11	2.3249814954	2.3244755495	2.3242983946	2.3241718322	2.3240705683
12	2.2411039069	2.2403756422	2.2401205957	2.2399383710	2.2397925621
13	2.1501327390	2.1491218654	2.1487677992	2.1485148121	2.1483123730
14	2.0525319298	2.0511755436	2.0507004388	2.0503609612	2.0500893090
15	1.9491021271	1.9473453911	1.9467301469	1.9462905650	1.9459388267
16	1.8412749628	1.8390939512	1.8383305329	1.8377852169	1.8373489546
17	1.7313287095	1.7287528296	1.7278520436	1.7272088742	1.7266944894
18	1.6211197497	1.6181491397	1.6171101886	1.6163683147	1.6157749535
19	1.5087252241	1.5051613795	1.5039130969	1.5030211632	1.5023074328
20	1.3932928320	1.3892535163	1.3878473579	1.3868455487	1.3860457098
21	1.3017067689	1.2990759483	1.2981689802	1.2975250575	1.2970121212
22	1.2168448262	1.2119789043	1.2102506705	1.2090084277	1.2080100205
23	1.0851477799	1.0780936974	1.0756160472	1.0738436305	1.0724241279
24	0.9405899007	0.9324803764	0.9296546344	0.9276407351	0.9260324630
25	0.8153073642	0.8091597645	0.8070654825	0.8055866576	0.8044134784
26	0.7155495554	0.7073568710	0.7044198556	0.7023000120	0.7005910223
27	0.5684568836	0.5561249358	0.5517835511	0.5486751696	0.5461842154
28	0.4088321911	0.3990141967	0.3960291926	0.3940483025	0.3925541393
29	0.3557864408	0.3491632463	0.3463888083	0.3442535040	0.3424531437
30	0.2113010907	0.1980011300	0.1940565838	0.1915338313	0.1897055209
31	0.1694486681	0.1650858647	0.1628889835	0.1610428359	0.1593885055
32	0.0760223149	0.0707665231	0.0690316941	0.0678093572	0.0668369969

(b) $e(p)=k^2-k^2(p)$

N1/N2	200/200	300/300	400/400	600/600
ET=	8.9040	7.7140	6.6490	8.3590
j	e(1)	e(2)	e(3)	e(4)
1	9.9E-08	4.5E-08	2.5E-08	1.1E-08
2	-1.7E-10	-7.4E-11	-4.2E-11	-1.9E-11
3	-7.9E-10	-3.5E-10	-2.0E-10	-8.7E-11
4	-1.9E-09	-8.3E-10	-4.7E-10	-2.1E-10
5	-3.5E-09	-1.6E-09	-8.7E-10	-3.9E-10
6	-6.6E-09	-2.9E-09	-1.7E-09	-7.3E-10
7	-1.3E-08	-5.6E-09	-3.2E-09	-1.4E-09
8	-2.3E-08	-1.0E-08	-5.7E-09	-2.5E-09
9	-3.8E-08	-1.7E-08	-9.6E-09	-4.3E-09
10	-6.1E-08	-2.7E-08	-1.5E-08	-6.7E-09
11	-9.1E-08	-4.0E-08	-2.3E-08	-1.0E-08
12	-1.3E-07	-5.8E-08	-3.3E-08	-1.5E-08
13	-1.8E-07	-8.1E-08	-4.6E-08	-2.0E-08
14	-2.4E-07	-1.1E-07	-6.1E-08	-2.7E-08
15	-3.2E-07	-1.4E-07	-7.9E-08	-3.5E-08
16	-3.9E-07	-1.7E-07	-9.8E-08	-4.4E-08
17	-4.6E-07	-2.1E-07	-1.2E-07	-5.1E-08
18	-5.3E-07	-2.4E-07	-1.3E-07	-5.9E-08
19	-6.4E-07	-2.9E-07	-1.6E-07	-7.1E-08
20	-7.2E-07	-3.2E-07	-1.8E-07	-8.0E-08
21	-4.7E-07	-2.1E-07	-1.2E-07	-5.1E-08
22	-8.8E-07	-4.0E-07	-2.2E-07	-1.0E-07
23	-1.3E-06	-5.7E-07	-3.2E-07	-1.4E-07
24	-1.5E-06	-6.4E-07	-3.6E-07	-1.6E-07
25	-1.1E-06	-4.7E-07	-2.7E-07	-1.2E-07
26	-1.5E-06	-6.8E-07	-3.8E-07	-1.7E-07
27	-2.2E-06	-9.9E-07	-5.6E-07	-2.5E-07
28	-1.6E-06	-6.5E-07	-3.5E-07	-1.5E-07
29	-1.3E-06	-6.7E-07	-3.9E-07	-1.8E-07
30	-2.2E-06	-8.3E-07	-4.4E-07	-1.8E-07
31	-1.0E-06	-5.7E-07	-3.5E-07	-1.7E-07
32	-9.2E-07	-3.9E-07	-2.2E-07	-9.7E-08

TABLE IV. (a) Richardson extrapolations for the Munk profile with an elastic bottom. (b) Errors.

(a) $K^2(p,\ldots,q)=10^4 \times k^2(p,\ldots,q)$

N1/N2	200/200	300/300	400/400	600/600	
ET=	8.9040	7.7140	6.6490	8.3590	
j	$K^2(1)$	$K^2(1,2)$	$K^2(1,2,3)$	$K^2(1,2,3,4)$	Exact
1	2.9038346179	2.9048123031	2.9048240319	2.9048240971	2.9048240973
2	2.7853806272	2.7853789520	2.7853789522	2.7853789522	2.7853789522
3	2.7442802497	2.7442723851	2.7442723862	2.7442723862	2.7442723862
4	2.7058505588	2.7058317827	2.7058317867	2.7058317867	2.7058317867
5	2.6688852743	2.6688502873	2.6688503038	2.6688503038	2.6688503038
6	2.6298754553	2.6298092827	2.6298093259	2.6298093260	2.6298093260
7	2.5846045345	2.5844773969	2.5844774624	2.5844774627	2.5844774627
8	2.5314592539	2.5312299479	2.5312300115	2.5312300119	2.5312300119
9	2.4703582714	2.4699736555	2.4699736802	2.4699736807	2.4699736807
10	2.4014708645	2.4008640395	2.4008639738	2.4008639743	2.4008639743
11	2.3249814954	2.3240707928	2.3240705678	2.3240705683	2.3240705683
12	2.2411039069	2.2397930304	2.2397925617	2.2397925621	2.2397925621
13	2.1501327390	2.1483131665	2.1483123727	2.1483123730	2.1483123730
14	2.0525319298	2.0500904346	2.0500893082	2.0500893090	2.0500893090
15	1.9491021271	1.9459400023	1.9459388241	1.9459388267	1.9459388267
16	1.8412749628	1.8373491419	1.8373489463	1.8373489545	1.8373489546
17	1.7313287095	1.7266921256	1.7266944784	1.7266944895	1.7266944894
18	1.6211197497	1.6157726518	1.6157749752	1.6157749536	1.6157749535
19	1.5087252241	1.5023103038	1.5023074482	1.5023074327	1.5023074328
20	1.3932928320	1.3860220637	1.3860452322	1.3860457078	1.3860457098
21	1.3017067689	1.2969712918	1.2970134074	1.2970121237	1.2970121212
22	1.2168448262	1.2080861668	1.2080094850	1.2080100189	1.2080100205
23	1.0851477799	1.0724504314	1.0724238520	1.0724241280	1.0724241279
24	0.9405899007	0.9259927570	0.9260311309	0.9260324557	0.9260324630
25	0.8153073642	0.8042416848	0.8044165506	0.8044135149	0.8044134784
26	0.7155495554	0.7008027234	0.7005906830	0.7005909854	0.7005910223
27	0.5684568836	0.5462593775	0.5461825685	0.5461842115	0.5461842154
28	0.4088321911	0.3911598012	0.3925351732	0.3925567401	0.3925541393
29	0.3557864408	0.3438646907	0.3424740012	0.3424505132	0.3424531437
30	0.2113010907	0.1873611615	0.1895263121	0.1897132708	0.1897055209
31	0.1694486681	0.1615956221	0.1595540218	0.1593804660	0.1593885055
32	0.0760223149	0.0665618896	0.0668809699	0.0668366602	0.0668369969

(b) $e(p,\ldots,q)=k^2-k^2(p,\ldots,q)$

N1/N2	200/200	300/300	400/400	600/600
ET=	8.9040	7.7140	6.6490	8.3590
j	e(1)	e(1,2)	e(1,2,3)	e(1,2,3,4)
1	9.9E-08	1.2E-09	6.5E-12	1.5E-14
2	-1.7E-10	2.0E-14	-6.4E-16	-7.3E-16
3	-7.9E-10	1.0E-13	2.8E-16	2.9E-16
4	-1.9E-09	3.9E-13	2.1E-16	-2.7E-16
5	-3.5E-09	1.6E-12	4.8E-15	-1.8E-16
6	-6.6E-09	4.3E-12	1.7E-14	-6.4E-16
7	-1.3E-08	6.6E-12	3.2E-14	-3.4E-16
8	-2.3E-08	6.4E-12	4.2E-14	-8.8E-16
9	-3.8E-08	2.5E-12	4.9E-14	7.4E-16
10	-6.1E-08	-6.5E-12	4.9E-14	4.2E-16
11	-9.1E-08	-2.2E-11	4.5E-14	8.0E-17
12	-1.3E-07	-4.7E-11	3.9E-14	4.5E-16
13	-1.8E-07	-7.9E-11	3.8E-14	6.8E-16
14	-2.4E-07	-1.1E-10	7.5E-14	3.2E-16
15	-3.2E-07	-1.2E-10	2.6E-13	1.8E-16
16	-3.9E-07	-1.9E-11	8.2E-13	4.4E-16
17	-4.6E-07	2.4E-10	1.1E-12	-4.7E-15
18	-5.3E-07	2.3E-10	-2.2E-12	-1.3E-14
19	-6.4E-07	-2.9E-10	-1.5E-12	1.9E-14
20	-7.2E-07	2.4E-09	4.8E-11	2.0E-13
21	-4.7E-07	4.1E-09	-1.3E-10	-2.5E-13
22	-8.8E-07	-7.6E-09	5.4E-11	1.6E-13
23	-1.3E-06	-2.6E-09	2.8E-11	-1.1E-14
24	-1.5E-06	4.0E-09	1.3E-10	7.3E-13
25	-1.1E-06	1.7E-08	-3.1E-10	-3.7E-12
26	-1.5E-06	-2.1E-08	3.4E-11	3.7E-12
27	-2.2E-06	-7.5E-09	1.6E-10	3.9E-13
28	-1.6E-06	1.4E-07	1.9E-09	-2.6E-10
29	-1.3E-06	-1.4E-07	-2.1E-09	2.6E-10
30	-2.2E-06	2.3E-07	1.8E-08	-7.7E-10
31	-1.0E-06	-2.2E-07	-1.7E-08	8.0E-10
32	-9.2E-07	2.8E-08	-4.4E-09	3.4E-11

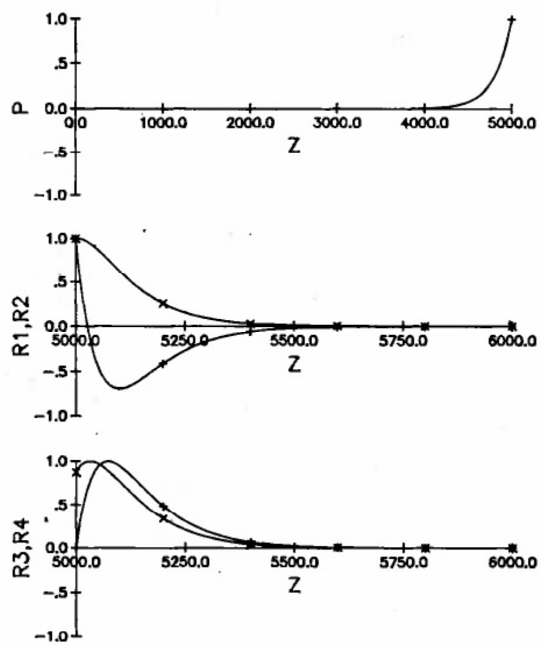

FIG. 4. Mode 1 for the Munk profile with an elastic bottom. $R_1 = 6.15E+3$ $r_1(+)$, $R_2 = 1.19E+5$ $r_2(\times)$, $R_3 = 2.24E-2$ $r_3(+)$, $R_4 = -8.72E-1$ $r_4(\times)$.

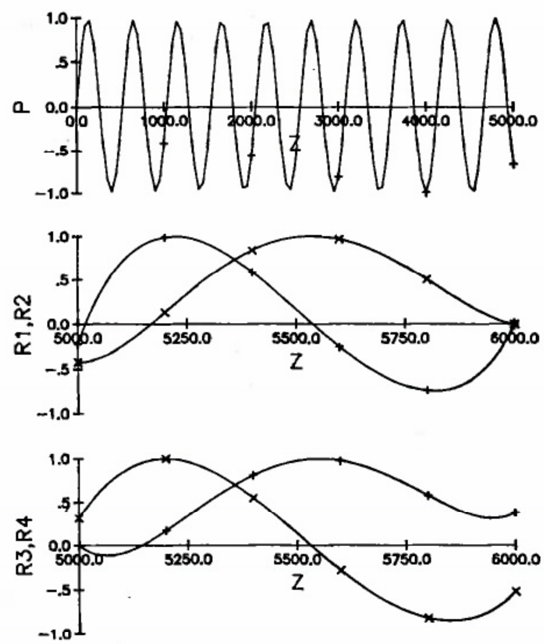

FIG. 6. Mode 21 for the Munk profile with an elastic bottom. $R_1 = 1.06E+3$ $r_1(+)$, $R_2 = 3.27E+4$ $r_2(\times)$, $R_3 = 4.45E-3$ $r_3(+)$, $R_4 = 4.66E-1$ $r_4(\times)$.

errors are reduced by as much as a factor of 1000 with each extrapolation. The quantity ET in these tables is the execution time required on the Northwestern Cyber 170/730 to compute all the given eigenvalues for the indicated mesh width. The row labeled "N_1/N_2" is the ratio of the number of subintervals in the ocean to the number of subintervals in the bottom. We observe that the execution times are *less* for the *finer* meshes, thus demonstrating the merit of using Richardson extrapolations to generate initial guesses.

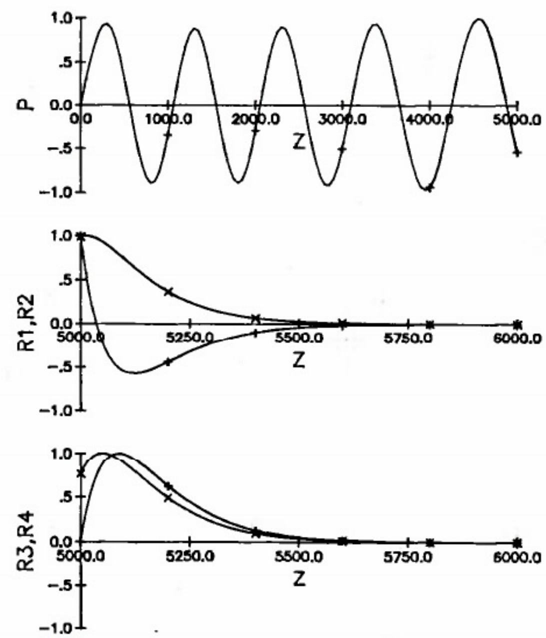

FIG. 5. Mode 10 for the Munk profile with an elastic bottom. $R_1 = -6.12E+3$ $r_1(+)$, $R_2 = 1.52E+5$ $r_2(\times)$, $R_3 = -2.73E-2$ $r_3(+)$, $R_4 = 1.41E+0$ $r_4(\times)$.

In Figs. 1–3 we have graphed the eigenfunctions for modes 3, 9, and 15. The scaling information is given in the figure captions. The vector (p, r_1, r_2, r_3, r_4) has been scaled such that $\max|p(z)| = 1$. These modes are representative of three classes of modes which exist for this problem. The first class, modes 1–7, are seismic modes, i.e., modes which have phase velocities less than the sound speed in the ocean and are consequently evanescent in the ocean. The second class of modes, modes 8–10, have a phase velocity greater than both the S-wave speed in the bottom and the sound speed in the ocean but less than the P-wave speed in the bottom. These modes are oscillatory in the ocean, but only the S-wave potential is oscillatory in the bottom, as we can show. The third class of modes, modes 11–18, have a phase speed greater than the S- or P-wave speeds in both the ocean and the bottom and are therefore oscillatory throughout the region.

The second example that we consider is an ocean with a Munk sound speed profile overlying a relatively rigid bottom with linearly increasing S and P wave speeds. The parameter values for the Munk profile are those previously employed,[25]

$$\omega = 8\pi/\text{s}, \quad D_1 = 5000\ \text{m}, \quad D_2 = 6000\ \text{m},$$
$$c(z) = 1500[1 + 0.000737(x - 1 + e^{-x})]\ \text{m/s}, \tag{20}$$
$$x = 2(z - 1300)/1300.$$

The P and S wave speeds are taken as

$$\begin{aligned} c_p(z) &= 4700 + 100(z - D_1)/(D_2 - D_1)\text{m/s}, \\ c_s(z) &= 2000 + 100(z - D_1)/(D_2 - D_1)\text{m/s}. \end{aligned} \tag{21}$$

TABLE V. Comparison of eigenvalues for the Munk profile with an elastic bottom and a rigid bottom.

j	Elastic	Rigid
1	0.29048241E-03	
2	0.27853790E-03	0.27853790E-03
3	0.27442724E-03	0.27442742E-03
4	0.27058318E-03	0.27059198E-03
5	0.26688503E-03	0.26704188E-03
6	0.26298093E-03	0.26397293E-03
7	0.25844775E-03	0.26073984E-03
8	0.25312300E-03	0.25635956E-03
9	0.24699737E-03	0.25098018E-03
10	0.24008640E-03	0.24474192E-03
11	0.23240706E-03	0.23768501E-03
12	0.22397926E-03	0.22982337E-03
13	0.21483124E-03	0.22116310E-03
14	0.20500893E-03	0.21170732E-03
15	0.19459388E-03	0.20145785E-03
16	0.18373490E-03	0.19041582E-03
17	0.17266945E-03	0.17858200E-03
18	0.16157750E-03	0.16595691E-03
19	0.15023074E-03	0.15254093E-03
20	0.13860457E-03	0.13833434E-03
21	0.12970121E-03	0.12333734E-03
22	0.12080100E-03	0.10755009E-03
23	0.10724241E-03	0.90972717E-04
24	0.92603246E-04	0.73605315E-04
25	0.80441348E-04	0.55447959E-04
26	0.70059102E-04	0.36500712E-04
27	0.54618422E-04	0.16763621E-04
28	0.39255414E-04	
29	0.34245314E-04	
30	0.18970552E-04	
31	0.15938851E-04	
32	0.66836997E-05	

The numerically obtained eigenvalues and their errors for typical meshes are given in Table III. Successive extrapolations and their errors are given in Table IV. The column labeled "exact" in Table III(a) was obtained by our program using higher-order extrapolation. For this problem there exist eight classes of modes including an interfacial mode characterized by evanescence away from the interface. Representatives of some of these classes of modes are graphed in Figs. 4–6.

The significance of the elastic bottom is illustrated in Table V in which we have presented the eigenvalues for this problem and the eigenvalues for a problem with the same sound speed profile in the ocean but with a rigid bottom. Naturally, there is no equivalent for the interface mode. Modes 2–6, which are trapped in the duct of the Munk profile are largely unaffected by the bottom model as expected. In contrast, the highest-order modes of the two models appear to be completely unrelated. These results demonstrate that with relatively rigid ocean bottoms, some energy is coupled into the bottom by modes which are not trapped in an ocean duct.

ACKNOWLEDGMENTS

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