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NUMERICAL METHOD FOR ACOUSTIC NORMAL MODES FOR SHEAR FLOWS†

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The normal modes and their propagation numbers for acoustic propagation in wave guides with flow are the eigenvectors and eigenvalues of a boundary value problem for a non-standard Sturm-Liouville problem. It is non-standard because it depends non-linearly on the eigenvalue parameter. (In the classical problem for ducts with no flow, the problem depends linearly on the eigenvalue parameter.) In this paper a method is presented for the fast numerical solution of this problem. It is a generalization of a method that was developed for the classical problem. A finite difference method is employed that combines well known numerical techniques and a generalization of the Sturm sequence method to solve the resulting algebraic eigenvalue problem. Then a modified Richardson extrapolation method is used that dramatically increases the accuracy of the computed eigenvalues. The method is then applied to two problems. They correspond to acoustic propagation in the ocean in the presence of a current, and to acoustic propagation in shear layers over flat plates.

1. INTRODUCTION

:e method of normal modes is a standard procedure for solving acoustic propagation oblems in ducts with flows and in shear layers; see, e.g., references [1, 2]. Typical amples of such problems are the propagation from a sound source in the ocean in the esence of a current, the propagation of sound through a duct of a jet engine, and the opagation in boundary layer or other shear flows over flat plates. In the last example, the flow is considered to extend to "infinity" normal to the plate, then the normal odes contribute to the more general spectral representation of the acoustic field. owever, flows over flat plates are frequently modeled by a finite layer with the pressure the upper surface of the layer taken as the far field pressure distribution.

The normal modes for the duct can be determined analytically only for relatively simple ws and sound speed distributions. More generally, it is necessary to determine them merically. In this paper, problems of two-dimensional stratified flows are considered, ch as boundary layer or other shear flows, through ducts of finite depth D, where the und speed of the fluid is also stratified. The appropriate normal mode problem for such ws is then reduced to an eigenvalue problem for a second-order, linear ordinary fferential equation. Since the acoustic medium is stratified, the coefficients in this uation depend only on the depth variable z. The eigenvalue parameter k is the opagation number of the acoustic waves and the eigenfunctions are the normal modes.

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Since k occurs non-linearly in this equation, it is a non-standard Sturm–Liouville eiger value problem unless the flow velocity is identically constant. There have been sever methods developed for the numerical solution of the standard problem; see, e.g., referenc [3] and references given therein. In particular, in reference [3] a fast finite differenc method to determine accurately the propagation numbers and the normal modes h been presented. The method is a combination of well-known numerical procedures suc as Sturm sequences, bisection, Brent's and Newton's methods for root finding, inver iteration, and finally a modified form of Richardson extrapolation. A minor modificatic of the method is required to apply it to the present problem with non-constant flov because of its non-linear dependence on k, for which the Sturm sequence procedu previously employed is not directly applicable.

The normal mode problem is formulated in section 2. The method and the requir modification are briefly outlined in section 3 and the Appendix. Finally, to demonstra the efficiency of the method it is applied to two problems in section 4. In the first proble a parabolic profile for the stratified flow velocity, to simulate an ocean current, considered. The Munk [4] profile is employed for the stratified ocean sound speed. the second problem the flow velocity is linearly stratified and the sound speed is constar thus simulating the flow in a shear layer over a flat plate.

2. FORMULATION

The two-dimensional basic flow velocity $\mathbf{U}(z)$, which is parallel to the rigid wall $z =$ is given by

$$\mathbf{U}(z) = (u^0(z), 0, 0), \qquad (2$$

where the horizontal flow velocity $u^0(z)$ is a specified function. The upper layer of t duct is at $z = 0$, so that the z co-ordinate is directed downward. The two-dimensior Euler's equations for the adiabatic flow of a compressible fluid are then linearized ab the steady stratified flow in equation (2.1) to obtain the acoustic equations. Then eliminating the density and the entropy from the resulting equations one can determi that the acoustic velocity vector $\mathbf{u}^*$, which has components $[u^*(x, z, t), 0, w^*(x, z, t)]$, a the reduced acoustic pressure $p^* = \rho_s P$, where P is the physical pressure, and ρ_s is t constant fluid density of this flow, satisfy a system of three partial differential equatio One seeks solutions of these equations in the form

$$u^*(x, z, t) = u(z)\, e^{i(kx-\omega t)}, \qquad w^*(x, z, t) = w(z)\, e^{i(kx-\omega t)},$$
$$p^*(x, z, t) = p(z)\, e^{i(kx-\omega t)}, \qquad (2$$

where ω is a specified radian frequency and the propagation numbers k are to determined. The depth dependent amplitudes then satisfy a system of three ordin differential equations. By eliminating $u(z)$ and $w(z)$ from these resulting equations, c can show that $p(z)$ satisfies the self-adjoint, second-order ordinary differential equat

$$(p'/a(z))' + b(z)p = 0, \qquad ($$

where primes denote differentiation with respect to z and the functions $a(z; \omega, k)$ b $b(z; \omega, k)$ are defined by

$$a(z; \omega, k) = (\omega - ku^0(z))^2, \qquad b(z; \omega, k) = 1/c^2(z) - k^2/a(z; \omega; k) \qquad ($$

and $c(z)$ is the specified stratified sound speed of the fluid.

To complete the formulation of the boundary value problem one specifies bounc conditions that on the upper surface of the duct $z = 0$ the pressure is a constant, wh is here set equal to zero. On the lower surface $z=D$ the acoustic normal velocity vanishes. This gives

$$p(0)=0, \qquad p'(D)/a(D)=0. \tag{2.5}$$

The boundary conditions in equations (2.5) are standard for ocean acoustics propagation. A brief discussion and interpretation of the pressure release condition at $z=0$ for the shear layer flow over a flat plate is given in section 4.

In summary, the eigenvalue problem is as follows: for specified shear flows $u^0(z)$, radian frequencies ω of the source and sound speeds $c(z)$, determine the propagation numbers $k=k_j$ for which equations (2.3) to (2.5) have non-trivial solutions (normal modes) $p_j(z)$. It is not a standard Sturm-Liouville eigenvalue problem since equation (2.3) depends non-linearly on the parameter k.

If the medium is stationary, so that $u^0(z)\equiv 0$, then equations (2.3) to (2.5) are reduced to

$$p''+(\omega^2/c^2(z)-k^2)p=0, \qquad p(0)=p'(D)=0, \tag{2.6}$$

which is a standard Sturm-Liouville eigenvalue problem since it depends linearly on the parameter k^2.

3. THE NUMERICAL METHOD

One first defines a mesh by dividing the interval $0<z<D$ into N equal subintervals by the points $z_i=ih$, $i=0,1,\ldots,N$, where the mesh width h is defined by $h=D/N$. Then one approximates the differential equation (2.3) on this mesh by replacing the first derivative in this self-adjoint operator by a standard two-point centered difference approximation, centered at the midpoint of the subinterval. This leads to the difference approximation

$$a_{i-1/2}^{-1}p_{i-1}-[a_{i-1/2}^{-1}+a_{i+1/2}^{-1}-h^2b_i]p_i+a_{i+1/2}^{-1}p_{i+1}=0, \qquad i=1,2,\ldots,N, \tag{3.1}$$

where the notation

$$a_{i+1/2}=a(z_{i+1/2}), \qquad b_i=b(z_i), \qquad i=0,1,\ldots,N \tag{3.2}$$

has been used and p_i, $i=0,1,\ldots,N$ are approximations to the eigenfunctions evaluated at the mesh points. The quantity p_{N+1} is the value of p corresponding to the fictitious point $z_{N+1}=D+h$. The boundary conditions (2.5) are approximated by

$$p_0=0, \qquad a_{N-1/2}^{-1}(p_N-p_{N-1})+a_{N+1/2}^{-1}(p_{N+1}-p_N)=0. \qquad (3.3, 3.4)$$

By solving equation (3.4) for p_{N+1} and substituting the result into equation (3.1) with $i=N$, one finally obtains the algebraic eigenvalue problem

$$A(k)\mathbf{p}=0 \tag{3.5}$$

as an approximation to the continuous eigenvalue problem in equations (2.3) to (2.5). Here $\mathbf{p}$ is the N-dimensional vector with components $p_1,\ldots,p_N$ and the tridiagonal $N\times N$ matrix A is defined by

$$A=\begin{bmatrix} -(a_{1/2}^{-1}+a_{3/2}^{-1})+h^2b_i & a_{3/2}^{-1} & \bigcirc \\ \cdots & \cdots & \cdots \\ a_{i-1/2}^{-1} & -(a_{i-1/2}^{-1}+a_{i+1/2}^{-1})+h^2b_i & a_{i+1/2}^{-1} \\ \cdots & \cdots & \cdots \\ \bigcirc & 2a_{N-1/2}^{-1} & -2a_{N-1/2}^{-1}+h^2b_N \end{bmatrix}. \tag{3.6}$$

The midpoint "averaged" difference approximation in expression (3.4) for the derivati yields a matrix with elements involving only values of $a(z)$ and $b(z)$ inside the doma: It is a non-linear algebraic eigenvalue problem in the sense that the elements of A(depend non-linearly on k.

The modified Richardson extrapolation procedure, which is now to be briefly describe is used to obtain more accurate estimates of the eigenvalues of the continuous proble in equations (2.3) to (2.5) from the eigenvalues $k_j(h)$ of the algebraic problem in equati (3.5). A discussion and comparison of the standard and modified Richardson extrapolati procedures for the eigenvalue problem (2.6) has been given in reference [3]. Thus, o expresses the jth eigenvalue $k_j(h)$ of equation (3.5) with mesh width h as

$$k_j(h) = k_j^I(0) - k_j^I(h) + \hat{k}_j + c_2 h^2 + c_4 h^4 \dots \quad (3$$

Here $k_j^I(h)$ is the jth eigenvalue of the algebraic eigenvalue problem (3.5) where t sound speed and convection velocities are replaced by "averaged" values $\bar{c}$ and $\bar{u}$ respectively. In addition, $k_j^I(0)$ corresponds to the exact eigenvalue of the continuc problem in equations (2.3) to (2.5), where $c(z)$ and $u^0(z)$ are replaced by the consta values $\bar{c}$ and $\bar{u}^0$, respectively. These isovelocity eigenvalues satisfy the dispersion relati

$$\{[\omega - k_j^I(h)\bar{u}^0]/\bar{c}\}^2 - [k_j^I(h)]^2 = E_j^2(h), \quad (3.8$$

where $E_j(h)$ is defined by

$$E_j(h) \equiv \sin\{[(j-\tfrac{1}{2})(\pi/D)](h/2)\}/(h/2), \quad (3.8$$

and $c_2, c_4, \dots$ are constants. The constant $\hat{k}_j$ is then the modified Richardson approximati to the jth eigenvalue of equations (2.3) to (2.5). It is determined from the linear algebr: system that results from applying expression (3.7) to a sequence of successively fir meshes $\{h_j\} = h_1, h_2, \dots, h_m$. Since the approximation depends on the sequence of me widths that is employed, it is appropriate to denote the approximation corresponding the meshes $h_p, h_{p+1}, \dots, h_{p+q}$ by $\hat{k}_j(p, \dots, p+q)$.

In brief, the values $k_j(h_p)$ for the sequence of mesh widths are obtained as follo For the coarsest mesh width, one first finds an isolating interval for each eigenvalue This is defined as an interval in k which contains only the eigenvalue k_j. For the first, largest, eigenvalue an upper bound for this interval is given by Theorem 1 in the Append Zero is taken as the lower bound since one is considering only modes which propaga to the right: i.e., in the downstream direction. This interval is successively bisected ur it contains only the first eigenvalue. In reference [3] this condition was determined the algebraic problem corresponding to equation (2.6) by counting sign changes in t Sturm sequence. Since the algebraic system of equation (3.5) is non-linear in the eigenval parameter, the Sturm sequence method is not directly applicable to the present proble In Theorem 2 of the Appendix an index function is derived which gives a zero-counti procedure for the present problem. It is a generalization of the Sturm counting meth as is discussed in the Appendix.

The process is repeated for each subsequent eigenvalue. Now, however, the previc eigenvalue's lower bound is an upper bound for the next eigenvalue. In addition, lo bounds for the current eigenvalue may have already been computed during the bisecti process for the previous eigenvalues. The isolating intervals provide initial estimates each eigenvalue. More accurate approximations of each eigenvalue are then obtained solving for the roots of the characteristic function by using Brent's method, see refere [5], which combines bisection, linear interpolation and inverse quadratic interpolati Convergence is then guaranteed to the isolated eigenvalue.

Initial guesses for the eigenvalues corresponding to the second and subsequent meshes are obtained by using the modified Richardson extrapolation procedure, but now extrapolating to the desired mesh size. Since isolating intervals are not obtained for these meshes, Brent's method is not applicable. One then uses the secant method to refine the root, although other procedures such as Newton's method could be employed, as was done in reference [3]. This change was motivated by the greater difficulty in computing the derivative required for Newton's method in the present problem. The choice of mesh sizes is motivated by convergence and speed of computation, as discussed in reference [3].

After the eigenvalues are obtained to the desired accuracy by the modified Richardson extrapolation procedure, the eigenfunctions are found by an inverse iteration, see reference [6], defined by

$$\mathbf{p}^0 = \mathbf{1}, \qquad A\mathbf{p}^{s+1} = \mathbf{p}^s, \qquad s = 1, 2, \ldots, \tag{3.9}$$

where the eigenvalues and difference equations of the final mesh are employed. (See reference [3] for more detailed discussions of the speed and accuracy of the method and comparisons with other methods for the numerical solution of equation (2.6).)

4. APPLICATIONS OF THE METHOD

Two applications to demonstrate the method are presented in this section. The first application, which is motivated by acoustic propagation in the deep ocean in the presence

TABLE 1(a)

Numerical eigenvalues $K_j(p) \equiv k_j(p) \times 10^2$ for the Munk profile with a shear flow

$N =$ $ET =$ j	77 0·5990 $K_j(1)$	92 0·2950 $K_j(2)$	115 0·3320 $K_j(3)$	146 0·3740 $K_j(4)$	"Refined" $\hat{K}_j(1, 2, \ldots, 6)$
1	1·6683910356	1·6683900317	1·6683891873	1·6683886180	1·6683876881
2	1·6561391651	1·6561344266	1·6561304410	1·6561277537	1·6561233648
3	1·6446288764	1·6446175231	1·6446079753	1·6446015384	1·6445910275
4	1·6339082100	1·6338892127	1·6338732459	1·6338624864	1·6338449252
5	1·6245773713	1·6245529575	1·6245324471	1·6245186302	1·6244960865
6	1·6146308702	1·6145837542	1·6145440886	1·6145173250	1·6144735835
7	1·6011227064	1·6010286247	1·6009493706	1·6008958698	1·6008083855
8	1·5844601893	1·5842931934	1·5841524299	1·5840573622	1·5839018298
9	1·5650044376	1·5647281245	1·5644950558	1·5643375649	1·5640797632
10	1·5428093778	1·5423750916	1·5420084983	1·5417606387	1·5413546617
11	1·5178372599	1·5171819688	1·5166283592	1·5162538167	1·5156399269
12	1·4900109981	1·4890545443	1·4882457513	1·4876981754	1·4867999970
13	1·4592255438	1·4578672391	1·4567174252	1·4559383404	1·4546593258
14	1·4253484436	1·4234628209	1·4218647112	1·4207808749	1·4189998084
15	1·3882160453	1·3856474504	1·3834674956	1·3819874827	1·3795526272
16	1·3476266485	1·3441821866	1·3412541901	1·3392638601	1·3359851289
17	1·3033303010	1·2987703811	1·2948868579	1·2922431553	1·2878813090
18	1·2550140114	1·2490392697	1·2439393376	1·2404615173	1·2347127099
19	1·2022800638	1·1945126259	1·1878643783	1·1833210798	1·1757938037
20	1·1446133203	1·1345688075	1·1259424751	1·1200317544	1·1102106641
21	1·0813299784	1·0683735441	1·0571983426	1·0495149799	1·0367004770
22	1·0114931761	0·9947660697	0·9802559212	0·9702338116	0·9534327835
23	0·9337648270	0·9120516611	0·8930654770	0·8798657902	0·8575717465
24	0·8461226163	0·8175847393	0·7923296454	0·7745922487	0·7442700035
25	0·7452526106	0·7068053509	0·6720821039	0·6472466223	0·6037941518
26	0·6249984014	0·5704642723	0·5190682563	0·4806930362	0·4090696885

TABLE 1(b)

Errors in numerical eigenvalues $e_j(p) \equiv k_j - k_j(p)$ for the Munk profile with a shear flow

$N=$	77	92	115	146
$ET=$	0·5990	0·2950	0·3320	0·3740
j	$e_j(1)$	$e_j(2)$	$e_j(3)$	$e_j(4)$
1	−3·3E−08	−2·3E−08	−1·5E−08	−9·3E−09
2	−1·6E−07	−1·1E−07	−7·1E−08	−4·4E−08
3	−3·8E−07	−2·6E−07	−1·7E−07	−1·1E−07
4	−6·3E−07	−4·4E−07	−2·8E−07	−1·8E−07
5	−8·1E−07	−5·7E−07	−3·6E−07	−2·3E−07
6	−1·6E−06	−1·1E−06	−7·1E−07	−4·4E−07
7	−3·1E−06	−2·2E−06	−1·4E−06	−8·7E−07
8	−5·6E−06	−3·9E−06	−2·5E−06	−1·6E−06
9	−9·2E−06	−6·5E−06	−4·2E−06	−2·6E−06
10	−1·5E−05	−1·0E−05	−6·5E−06	−4·1E−06
11	−2·2E−05	−1·5E−05	−9·9E−06	−6·1E−06
12	−3·2E−05	−2·3E−05	−1·4E−05	−9·0E−06
13	−4·6E−05	−3·2E−05	−2·1E−05	−1·3E−05
14	−6·3E−05	−4·5E−05	−2·9E−05	−1·8E−05
15	−8·7E−05	−6·1E−05	−3·9E−05	−2·4E−05
16	−1·2E−04	−8·2E−05	−5·3E−05	−3·3E−05
17	−1·5E−04	−1·1E−04	−7·0E−05	−4·4E−05
18	−2·0E−04	−1·4E−04	−9·2E−05	−5·7E−05
19	−2·6E−04	−1·9E−04	−1·2E−04	−7·5E−05
20	−3·4E−04	−2·4E−04	−1·6E−04	−9·8E−05
21	−4·5E−04	−3·2E−04	−2·0E−04	−1·3E−04
22	−5·8E−04	−4·1E−04	−2·7E−04	−1·7E−04
23	−7·6E−04	−5·4E−04	−3·5E−04	−2·2E−04
24	−1·0E−03	−7·3E−04	−4·8E−04	−3·0E−04
25	−1·4E−03	−1·0E−03	−6·8E−04	−4·3E−04
26	−2·2E−03	−1·6E−03	−1·1E−03	−7·2E−04

of a current, is called the ocean acoustics problem. One considers a parabolic profile for the current $u^0(z)$ and a Munk profile for the stratified sound speed $c(z)$, using the parameters suggested in reference [7] and employed in reference [3]. Specifically, one employs the following parameters: $\omega = 8\pi$, $D = 5000$, $u^0(z) = \frac{3}{2}[(z-D)/D]^2$, $c(z) = 1500[1\cdot0 + 0\cdot00737(x - 1 + e^{-x})]$, where the parameter $x(z)$ is defined by $x = 2(z-D)/D$ and lengths and times are measured in metres and seconds, respectively. The Mach number for this current is approximately 1/1000. In Table 1(a) the numerically determined eigenvalues of the algebraic problem for the indicated mesh widths ($h = D/N$) and for the downstream traveling modes are presented. The column labeled "refined" contains the eigenvalues numerically determined by the method of using extrapolations with several more refined meshes. The errors in these eigenvalues, which are defined as the differences between them and the "refined" eigenvalues are shown in Table 1(b). In Table 2 the eigenvalues obtained by the modified Richardson extrapolation procedure, and their errors, are presented.

One can observe from Table 1 the anticipated $O(h^2)$ convergence in that doubling the number of mesh points reduces the error by a factor of about 4. In contrast, the errors in the eigenvalues obtained from the extrapolation method are reduced by as much as a factor of 1000 with each additional extrapolation, for the first few eigenvalues. This factor decreases for the higher order modes. The symbol ET in the tables denotes the execution time in seconds on the Northwestern Cyber 170/730 to compute all the given eigenvalues corresponding to each mesh. One can observe that the execution times for the finer meshes are less than the execution times for the coarsest mesh. This demonstrates the merit of using Richardson extrapolations to generate initial guesses.

These results show that the efficiencies of the methods for the present problem and the stationary medium problem treated in reference [3] are comparable, with one significant difference: the extrapolation process for the present problem is less effective for the modes close to cut-off. This occurs even with $u^0 = 0$ in the present problem. This discrepancy is a result of extrapolating with k rather than with k^2, as was done in reference [3]. Intuitively, this occurs because the values $k^2(h)$ move smoothly along the real line as h is refined but the values of $k(h)$ lie along either the real or the imaginary axes, and a pair of eigenvalues on the real line can coalesce at the origin and then split along the imaginary axis as h decreases. The extrapolation process cannot determine this abrupt transition because real values of k will always yield real extrapolates. Although none of the eigenvalues shown in Table 2 actually split, it can be shown that proximity to the splitting point is sufficient to adversely effect the convergence.

In Table 3 the eigenvalues from this convected problem are compared to the eigenvalues obtained when $u^0 = 0$. The change varies from $2\cdot3 \times 10^{-6}$ to $5\cdot6 \times 10^{-6}$, an effect that would be significant at ranges of approximately 300 km and beyond. A reasonable estimate of

TABLE 2(a)

Modified extrapolations $\hat{K}_j(p, \ldots, q) \equiv \hat{k}_j(p, \ldots, q) \times 10^2$ *for the Munk profile with a shear flow*

$N =$ $ET =$ j	77 0·5990 $\hat{K}_j(1)$	92 0·2950 $\hat{K}_j(1, 2)$	115 0·3320 $\hat{K}_j(1, 2, 3)$	146 0·3740 $\hat{K}_j(1, 2, 3, 4)$	"Refined" $\hat{K}_j(1, 2, \ldots, 6)$
1	1·6683910253	1·6683876838	1·6683876881	1·6683876880	1·6683876881
2	1·6561383254	1·6561233440	1·6561233648	1·6561233648	1·6561233648
3	1·6446223804	1·6445909679	1·6445910274	1·6445910275	1·6445910275
4	1·6338831535	1·6338447688	1·6338449253	1·6338449252	1·6338449252
5	1·6245085270	1·6244958020	1·6244960873	1·6244960865	1·6244960865
6	1·6144761774	1·6144733694	1·6144735844	1·6144735835	1·6144735835
7	1·6008183841	1·6008080613	1·6008083870	1·6008083855	1·6008083855
8	1·5839153345	1·5839013570	1·5839018327	1·5839018298	1·5839018298
9	1·5640949334	1·5640791280	1·5640797684	1·5640797632	1·5640797632
10	1·5413708831	1·5413538417	1·5413546700	1·5413546616	1·5413546617
11	1·5156569760	1·5156388948	1·5156399399	1·5156399268	1·5156399269
12	1·4868177918	1·4867987209	1·4868000164	1·4867999969	1·4867999970
13	1·4546778378	1·4546577692	1·4546593540	1·4546593256	1·4546593258
14	1·4190190244	1·4189979290	1·4189998480	1·4189998080	1·4189998084
15	1·3795725261	1·3795503753	1·3795526817	1·3795526265	1·3795526272
16	1·3360056617	1·3359824453	1·3359852024	1·3359851278	1·3359851289
17	1·2879023731	1·2878781206	1·2878814063	1·2878813073	1·2878813090
18	1·2347341097	1·2347089234	1·2347128367	1·2347127073	1·2347127099
19	1·1758151826	1·1757892933	1·1757939657	1·1757937999	1·1757938037
20	2·1102313793	1·1102052468	1·1102108660	1·1102106585	1·1102106641
21	1·0367193521	1·0366938584	1·0367007176	1·0367004686	1·0367004770
22	0·9534475727	0·9534244224	0·9534330408	0·9534327696	0·9534327835
23	0·8575778368	0·8575604638	0·8575719198	0·8575717185	0·8575717465
24	0·7442566961	0·7442525985	0·7442696390	0·7442699161	0·7442700035
25	0·6037307156	0·6037587054	0·6037905600	0·6037936151	0·6037941518
26	0·4088149578	0·4089357245	0·4090310095	0·4090601306	0·4090696885

TABLE 2(b)

Errors in modified extrapolations $\hat{e}_j(p, \ldots, q) \equiv k_j - \hat{k}_j(p, \ldots, q)$ for the Munk profile with a shear flow

$N=$	77	92	115	146
$ET=$	0·5990	0·2950	0·3320	0·3740
j	$\hat{e}_j(1)$	$\hat{e}_j(1,2)$	$\hat{e}_j(1,2,3)$	$\hat{e}_j(1,2,3,4)$
1	−3·3E−03	4·2E−11	−7·1E−14	9·1E−14
2	−1·5E−07	2·1E−10	3·6E−13	4·1E−15
3	−3·1E−07	6·0E−10	4·4E−13	−1·6E−13
4	−3·8E−07	1·6E−09	−1·2E−12	9·2E−15
5	−1·2E−07	2·8E−09	−7·3E−12	1·4E−14
6	−2·6E−08	2·1E−09	−8·7E−12	2·8E−14
7	−1·0E−07	3·2E−09	−1·5E−11	4·5E−14
8	−1·4E−07	4·7E−09	−2·9E−11	1·1E−13
9	−1·5E−07	6·4E−09	−5·1E−11	2·3E−13
10	−1·6E−07	8·2E−09	−8·4E−11	4·5E−13
11	−1·7E−07	1·0E−08	−1·3E−10	7·9E−13
12	−1·8E−07	1·3E−08	−1·9E−10	1·4E−12
13	−1·9E−07	1·6E−08	−2·8E−10	2·7E−12
14	−1·9E−07	1·9E−08	−4·0E−10	4·4E−12
15	−2·0E−07	2·3E−08	−5·4E−10	7·1E−12
16	−2·1E−07	2·7E−08	−7·3E−10	1·1E−11
17	−2·1E−07	3·2E−08	−9·7E−10	1·7E−11
18	−2·1E−07	3·8E−08	−1·3E−09	2·5E−11
19	−2·1E−07	4·5E−08	−1·6E−09	3·7E−11
20	−2·1E−07	5·4E−08	−2·0E−09	5·6E−11
21	−1·9E−07	6·6E−08	−2·4E−09	8·4E−11
22	−1·5E−07	8·4E−08	−2·6E−09	1·4E−10
23	−6·1E−08	1·1E−07	−1·7E−09	2·8E−10
24	1·3E−07	1·7E−07	3·6E−09	8·7E−10
25	6·3E−07	3·5E−07	3·6E−08	5·4E−09
26	2·5E−06	1·3E−06	3·9E−07	9·6E−08

this effect of slow convective flows can be obtained by comparison to a related problem with the sound speed and convection velocities replaced by their average values, but we do not present these results.

In the second application of the method, which we call the aeroacoustic problem, the following parameters are employed: $\omega \equiv 3300\pi$, $D \equiv 1$, $u^0(z) \equiv 165(z-D)/D$, $c \equiv 330$. Then the corresponding dimensionless acoustic wave number $\equiv \omega D/c = 10\pi$ and the Mach number of the flow equals $\frac{1}{2}$. These parameters correspond to acoustic propagation at moderate subsonic flow velocities in a strongly sheared layer over the rigid surface $z = D$, where the "upper surface" of the layer is approximated by a constant pressure surface. Since the fluid above the sheared layer is considered to extend to infinity normal to the plate, the imposed boundary condition implies that one is considering only trapped modes.

In Table 4, the numerically determined eigenvalues and their errors are presented for a sequence of relatively coarse meshes. The modified Richardson extrapolates and their errors are presented in Table 5. The qualitative features of the results are similar to those in the previous problem: that is, the modified Richardson extrapolation method yields substantially more accurate eigenvalues.

Graphs of the eigenfunctions corresponding to the first nine eigenvalues are presented in Figure 1. The lower order modes, Modes 1 to 5, are trapped near the wall, $z = 1{\cdot}0$,

TABLE 3

Comparison of eigenvalues for Munk profile with and without a shear flow

j	No flow	Flow	Change
1	0·0166895	0·0166839	0·0000056
2	0·0165659	0·0165612	0·0000046
3	0·0164497	0·0164459	0·0000038
4	0·0163414	0·0163385	0·0000030
5	0·0162472	0·0162450	0·0000023
6	0·0161474	0·0161447	0·0000027
7	0·0160112	0·0160081	0·0000031
8	0·0158424	0·0158390	0·0000033
9	0·0156442	0·0156408	0·0000034
10	0·0154170	0·0154135	0·0000035
11	0·0151599	0·0151564	0·0000035
12	0·0148716	0·0148680	0·0000036
13	0·0145502	0·0145466	0·0000036
14	0·0141936	0·0141900	0·0000036
15	0·0137991	0·0137955	0·0000036
16	0·0133635	0·0133599	0·0000036
17	0·0128824	0·0128788	0·0000036
18	0·0123507	0·0123471	0·0000036
19	0·0117616	0·0117579	0·0000036
20	0·0111057	0·0111021	0·0000036
21	0·0103706	0·0103670	0·0000036
22	0·0095380	0·0095343	0·0000036
23	0·0085794	0·0085757	0·0000036
24	0·0074463	0·0074427	0·0000036
25	0·0060416	0·0060379	0·0000036
26	0·0040943	0·0040907	0·0000036

while the higher order modes, modes 6 to 9, are oscillatory throughout the interval and hence convey energy to the surface, $z = 0$.

In addition, the method has been applied to the aeroacoustic problem with a Mach number of 0·1. The first six modes are graphed in Figure 2. The shear flow is less effective

TABLE 4(a)

Numerical eigenvalues $K_j(p) \equiv k_j(p) \times 10^{-1}$ *for the aeroacoustic problem*

$N =$ $ET =$ j	29 0·1130 $K_j(1)$	34 0·0600 $K_j(2)$	43 0·0640 $K_j(3)$	55 0·0780 $K_j(4)$	"Refined" $\hat{K}_j(1, 2, \ldots, 7)$
1	2·9791779468	2·9786591353	2·9781475969	2·9778198593	2·9773107935
2	2·6652423256	2·6644571992	2·6636822541	2·6631851931	2·6624120617
3	2·4622206634	2·4608812904	2·4595613407	2·4587157468	2·4574020786
4	2·3007170774	2·2988227138	2·2969593731	2·2957674391	2·2939183407
5	2·1574305739	2·1546426081	1·1518887423	2·1501207769	2·1473676297
6	1·9976917863	1·9925412298	1·9874066359	1·9840851629	1·9788730950
7	1·7931851971	1·7832139036	1·7731929427	1·7666673184	1·7563582522
8	1·5320342062	1·5132312801	1·4941512707	1·4816262128	1·4616762399
9	1·1987109449	1·1635112361	1·1272494549	1·1031370029	1·0642062148
10	0·7561207089	0·6866601449	0·6125803937	0·5617222139	0·4765135425

TABLE 4(b)

Errors in numerical eigenvalues $e_j(p) \equiv k_j - k_j(p)$ for the aeroacoustic problem

$N=$	29	34	43	55
$ET=$	0·1130	0·0600	0·0640	0·0780
j	$e_j(1)$	$e_j(2)$	$e_j(3)$	$e_j(4)$
1	−1·9E−03	−1·3E−02	−8·4E−03	−5·1E−03
2	−2·8E−02	−2·0E−02	−1·3E−02	−7·7E−03
3	−4·8E−02	−3·5E−02	−2·2E−02	−1·3E−02
4	−6·8E−02	−4·9E−02	−3·0E−02	−1·8E−02
5	−1·0E−01	−7·3E−02	−4·5E−02	−2·8E−02
6	−1·9E−01	−1·4E−01	−8·5E−02	−5·2E−02
7	−3·7E−01	−2·7E−01	−1·7E−01	−1·0E−01
8	−7·0E−01	−5·2E−01	−3·2E−01	−2·0E−01
9	−1·3E+00	−9·9E−01	−6·3E−01	−3·9E−01
10	−2·8E+00	−2·1E+00	−1·4E+00	−8·5E−01

TABLE 5(a)

Modified extrapolations $\hat{K}_j(p, \ldots, q) \equiv \hat{k}_j(p, \ldots, q) \times 10^{-1}$ for the aeroacoustic problem

$N=$	29	34	43	55	
$ET=$	0·1130	0·0600	0·0640	0·0780	"Refined"
j	$\hat{K}_j(1)$	$\hat{K}_j(1,2)$	$\hat{K}_j(1,2,3)$	$\hat{K}_j(1,2,3,4)$	$\hat{K}_j(1,2,\ldots,7)$
1	2·9791769857	2·9772739907	2·9773112352	2·9773107932	2·9773107935
2	2·6651637893	2·6623609853	2·6624133592	2·6624120366	2·6624120617
3	2·4616036823	2·4573042366	2·4574046920	2·4574020189	2·4574020786
4	2·2982788039	2·2937558728	2·2939228992	2·2939182293	2·2939183407
5	2·1504915721	2·1471533640	2·1473748036	2·1473674436	2·1473676297
6	1·9813120748	1·9786146653	1·9788836762	1·9788727919	1·9788730950
7	1·7586553227	1·7560117723	1·7563755504	1·7563577060	1·7563582522
8	1·4637175150	1·4612162097	1·4617039503	1·4616752223	1·4616762399
9	1·0655171180	1·0635790669	1·0642442928	1·0642042047	1·0642062148
10	0·4749251844	0·4751279471	0·4764553170	0·4764974279	0·4765135425

TABLE 5(b)

Errors in modified extrapolations $\hat{e}_j(p, \ldots, q) \equiv k_j - \hat{k}_j(p, \ldots, q)$ for the aeroacoustic problem

$N=$	29	34	43	55
$ET=$	0·1130	0·0600	0·0640	0·0780
j	$\hat{e}_j(1)$	$\hat{e}_j(1,2)$	$\hat{e}_j(1,2,3)$	$\hat{e}_j(1,2,3,4)$
1	−1·9E−02	3·7E−04	−4·4E−06	3·3E−09
2	−2·8E−02	5·1E−04	−1·3E−05	2·5E−07
3	−4·2E−02	9·8E−04	−2·6E−05	6·0E−07
4	−4·4E−02	1·6E−03	−4·6E−05	1·1E−06
5	−3·1E−02	2·1E−03	−7·2E−05	1·9E−06
6	−2·4E−02	2·6E−03	−1·1E−04	3·0E−06
7	−2·3E−02	3·5E−03	−1·7E−04	5·5E−06
8	−2·0E−02	4·6E−03	−2·8E−04	1·0E−05
9	−1·3E−02	6·3E−03	−3·8E−04	2·0E−05
10	1·6E−02	1·4E−02	5·8E−04	1·6E−04

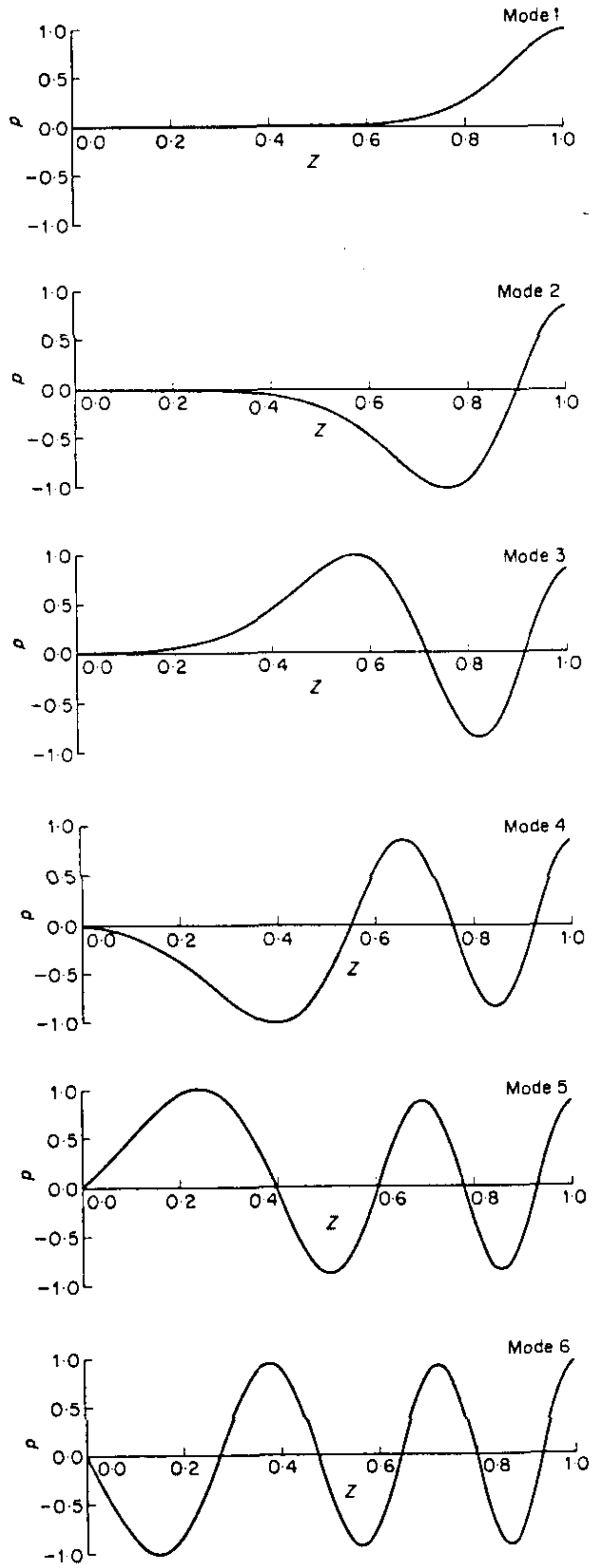

Figure 1. Graphs of the first nine modes for the aeroacoustic problem with $M = \frac{1}{2}$.

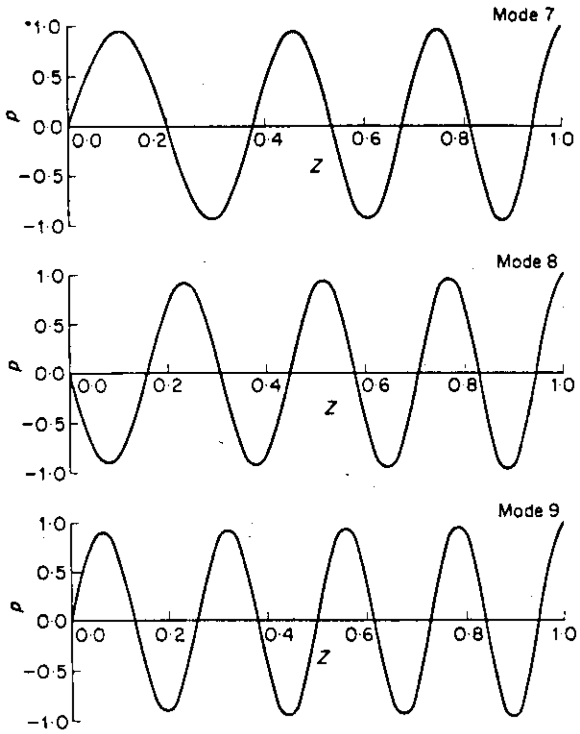

Figure 1. (*cont.*)

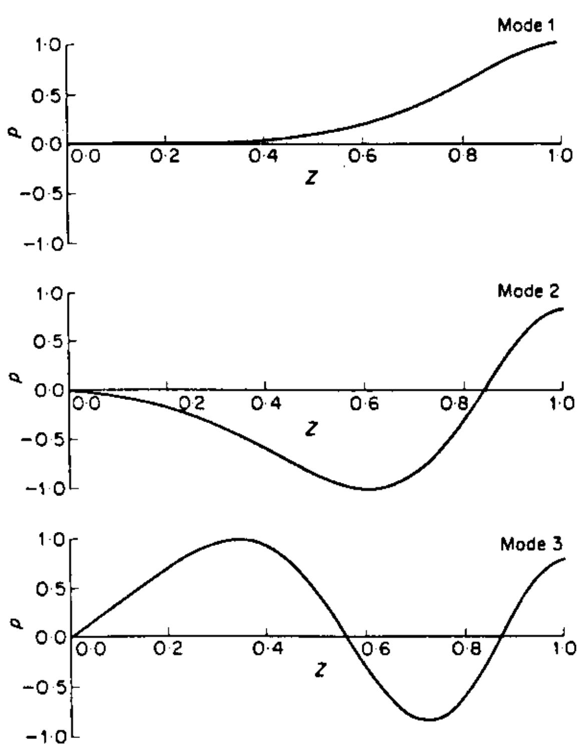

Figure 2. Graphs of the first six modes for the aeroacoustic problem with $M = \frac{1}{10}$.

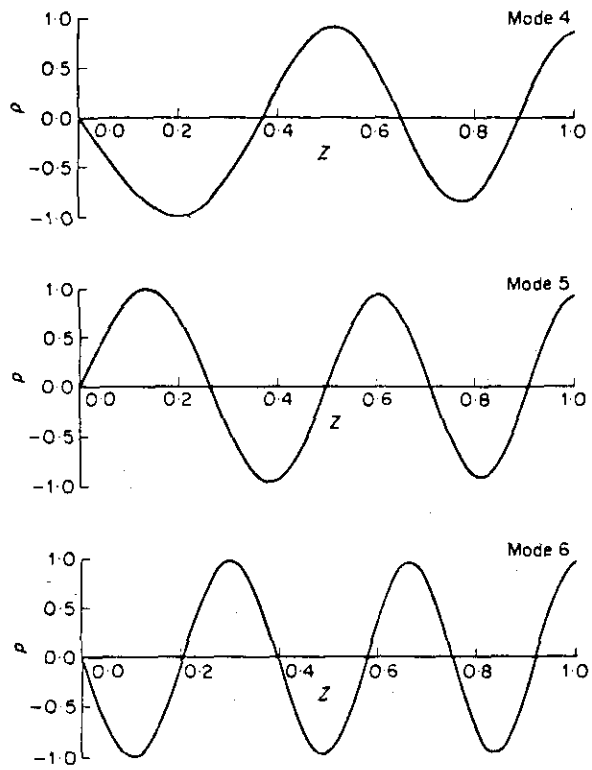

Figure 2. (*cont.*)

n forming an acoustic duct at this lower Mach number, as can be observed by comparing igures 1 and 2. Thus, more energy is carried to the free surface $z = 0$ at lower values of he Mach number for specfic modes.

With suitable modifications the present method can be extended to study acoustic ropagation in ducts where the rigid boundary at $z = D$ can be replaced by an impedance ondition or by an elastic layer, as has been demonstrated in reference [8].

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APPENDIX

In this Appendix certain mathematical results are established that are employed in obtaining the isolating intervals for the eigenvalues corresponding to the coarsest mesh size. Only subsonic flows are considered, so that max $(u^0(z)) < \min(c(z))$. The required upper bound for the largest eigenvalue is obtained by application of the following Theorem.

Theorem 1. The real eigenvalues k of equation (2.3) satisfy either

$$\omega/k > \min_{0\leq z\leq D}(u^0(z)+c(z)) \quad \text{or} \quad \omega/k < \max_{0\leq z\leq D}(u^0(z)-c(z)). \tag{A1}$$

Proof. Multiply both sides of the differential equation (2.3) by p and integrate the result from $z=0$ to $z=D$. Integration by parts is valid, as one can show. Then using the boundary conditions one gets

$$-\int_0^D (p'(z))^2/a(z)\,dz + \int_0^D b(z)(p(z))^2\,dz = 0. \tag{A2}$$

Since $a(z) \geq 0$ it follows from equation (A2) that there exists an interval in $(0, D)$ in which $b(z) > 0$. Therefore, in this interval one has, from the definition of $b(z)$ in equation (2.4), that $c^2 < (\omega/k - u^0)^2$, from which the desired result follows.

The theorem states that the bound on the phase velocity is increased or decreased by the flow velocity, depending on whether the wave is traveling upstream or downstream.

The index function which gives the zero-counting procedure to determine the isolating intervals is provided by the second theorem.

Theorem 2. For $\omega > 0$ and $k > 0$, the number of eigenvalues greater than k is obtained by integrating equation (2.3) with the initial conditions $p(0)=0$, $p'(0)=1$, and then calculating the index function $I(k)$, which is defined by

$$I(k) \equiv \text{the number of zeros of } p(z) \text{ in } (0, D] + \begin{Bmatrix} 1, \text{ if } p(D)p'(D) < 0 \\ 0, \text{ if } p(D)p'(D) \geq 0 \end{Bmatrix}. \tag{A3}$$

To prove this theorem one first requires the following oscillation results for the solutions of equation (2.3).

Lemma 1. Let $p(z)$ and $P(z)$ satisfy the initial value problems

$$\begin{aligned} (p'/a)' + bp &= 0, \quad p(0)=0, \quad p'(0)=1, \\ (P'/A)' + BP &= 0, \quad P(0)=0, \quad P'(0)=1, \end{aligned} \tag{A4}$$

in which the continuous coefficients satisfy the conditions $A > a > 0$ and $B > b$ for all z. The Prufer variables, $r(z)$, $R(z)$, $t(z)$ and $T(z)$ are defined by

$$\begin{aligned} p(z) &= r(z)\sin t(z), \quad & p'(z) &= r(z)\cos t(z), \\ P(z) &= R(z)\sin T(z), \quad & P'(z) &= R(z)\cos T(z). \end{aligned} \tag{A5}$$

Then the phases satisfy the inequality

$$T(z) > t(z), \quad \text{for } z > 0. \tag{A6}$$

Since this lemma is a trivial extension of an existing oscillation theorem, (see, e.g., reference [9]) we omit the proof. It can also be shown that $r(z)$ is positive and so the zeros of $p(z)$ ($p'(z)$) occur when the phase function, t, is an odd (even) multiple of $\pi/2$.

Proof of Theorem 2. One first can establish that $I(k_{max})=0$, where k_{max} is the upper bound on the eigenvalues implied by Theorem 1. After multiplying equation (2.3) by p and integrating from 0 to z one obtains

$$p(z)p'(z)/a(z)\Big|_0^z - \int_0^z (p')^2/a(z)\,dz + \int_0^z p^2 b(z)\,dz = 0. \tag{A7}$$

When $k=k_{max}$ $b(z)$ is negative and so one must have $p(z)p'(z)>0$. Hence, p has no zeros and $I(k_{max})=0$. As k is decreased, $a(z;k)$ and $b(z;k)$ both increase. It follows from the lemma that the phase function $t(z;k)$ increases as k is decreased. Since the eigenvalues occur at points where $t(d;k)$ is an odd multiple of $\pi/2$ and $0<t(D;k_{max})<\pi/2$, the number of eigenvalues greater than k is obtained as the largest n for which $t(D;k)>(2n-1)\pi/2$. Thus, the problem of determining the number of eigenvalues less than k is reduced to that of determining the number of rotations in the phase function. As the phase function is not to be computed explicitly, one needs to be able to count rotations in $p(z)$. If for some $z_0 t = m\pi$ then

$$t'(z_0) = a(z_0)\cos^2(m\pi) + b(z_0)\sin^2(m\pi) = a(z_0) > 0. \tag{A8}$$

This implies that $t(z)$ can cross through a line $t=m\pi$ at most once during the integration from $z=0$. Thus m, the number of whole multiples of π contained in $t(D)$, can be obtained by counting the zeros of $p(z)$. Now, if $2m\pi/2 \leq t(D;k) \leq (2m+1)\pi/2$, then n, the number of eigenvalues greater than k, will be equal to m, but if $(2m+1)\pi/2 < t(D;k) < (2m+2)\pi/2$, then n is equal to $m+1$. Clearly, the former case occurs when $p(D)p'(D)>0$ and the latter when $p(D)p'(D)<0$, thus establishing the theorem. This theorem may be extended to other quadrants of the $\omega-k$ plane with minor modifications.

To show the relationship between the index function of Theorem 2 and the standard Sturm sequence procedure, one can observe that a characteristic function† for the matrix may be obtained by setting $p_1=1$ and using the first $N-1$ of the difference equations (3.1) to recursively generate p_i, $i=1,\ldots,N$. The residual $r(k)$, which is defined by $r(k) \equiv 2a^{-1}_{N-1/2}p_{N-1} + (-2a^{-1}_{N-1/2} + h^2 b_N)p_N$, is then a characteristic function of A. This process has an obvious interpretation as an integration of system (2.3)-(2.5); thus the computation of $r(k)$ simultaneously generates the information required to compute the index function of Theorem 2. In addition, it is easy to verify that the p_i differ from the principal minors of $-A$ by products of the squares of the off-diagonal elements [6]. It follows that the index function gives the number of sign changes in the principal minors of $-A$, and that the present method is a generalization of the Sturm sequence method previously employed [3].

† A characteristic function of $A(k)$ is a function with zeros at the eigenvalues of A.