NUMERICAL METHODS OF ACOUSTIC MODELING

from PROCEEDINGS OF THE UNDERSEA DEFENSE TECHNOLOGY CONF. (LONDON) (1988)

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ABSTRACT

Computer modeling of the acoustic field in the ocean is at present accomplished using principally four types of models: 1) ray tracing, 2) normal modes, 3) FFP (fast-field program) and 4) the PE (parabolic equation). The diversity of models stems from the need to efficiently cover low to high frequencies, range-independent and range-dependent environments, and acoustic vs. elastic bottom types. In general, the choice of model involves trade-offs between accuracy, run time and ease of use. Present efforts are leading to models which provide full three-dimensional images of the acoustic field including effects of bottom variation due to seamounts, continental slope, etc., as well as oceanographic features such as fronts and eddies. While in the past most model work has been done for tonal (narrowband) signals, models which provide time-domain results for broadband and transient signals are also emerging. We briefly survey the various models and their present capabilities.

INTRODUCTION

Ocean acoustic models have been a primary beneficiary of the exponential increase in computing power which has occurred over the last 20 years or so. The early ray models have evolved from treating simple stratified environments to where they are now routinely used for range-dependent problems with bathymetric features such as seamounts and continental slopes as well as oceanographic features such as fronts and eddies. Normal mode models were originally constructed for simple two-layer environments (the Pekeris waveguide) and modified to allow arbitrary numbers of layers.

A useful starting point for underwater acoustics problems is the Helmholtz equation (or reduced wave equation). In two-dimensions it reads as follows

$$\nabla^2 p + \frac{\omega^2}{c^2(r,z)} p = \frac{-\delta(r - r_s)\delta(z - z_s)}{r}, \tag{1}$$

where $c(r, z)$ is the ocean sound speed as a function of range and depth. In addition, ω is the circular frequency of the source which is located at the range/depth coordinate (r_s, z_s). The object is to solve for the response of the channel to the source, that is to solve for the acoustic pressure $p(r, z)$.

In principle it is straightforward to solve such an equation using, for instance, standard finite difference techniques. In practice, this is almost never done. The finite difference schemes require 6-10 points per wavelength so that a modest problem of 50 km in range and 5 km in depth would require a grid of around 10^6 points for a 10 Hz source. By present day computer standards this problem is completely intractable.

For this reason a number of approximations have been introduced leading to principally four different kinds of models: 1) Ray tracing, 2) FFP (fast field program) 3) Normal mode, and 4) PE (parabolic equation). In the following sections we will briefly describe each of these approaches in terms of both their mathematical basis and domain of applicability.

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RAY THEORY

Ray-based models have been used for many years in underwater acoustics. In the early 60's virtually all modeling was done using either normal modes or ray tracing and primarily the latter. Today, however, ray tracing codes seem to have fallen somewhat out of favor in the research community. The problem being that the inherent (high frequency) approximation of the method leads to somewhat coarse accuracy in the results. On the other hand, ray methods still enjoy a strong following in the operational environment where speed is a critical factor, and environmental uncertainty poses much more severe constraints on the attainable accuracy.

To obtain the ray equations, one seeks a solution of the Helmholtz equation of the following form:

$$p(r,z) = e^{ik\phi(r,z)} \sum_{j=1}^{\infty} A_j(r,z) \frac{1}{(ik)^j}, \tag{2}$$

where $k = \omega/c_0$ and c_0 is a reference sound speed. Substituting into the Helmholtz equation one obtains an infinite sequence of equations for the functions $\phi(r,z)$ and $A_j(r,z)$:

$$(\nabla\phi)^2 = -c_0^2/c^2(r,z), \tag{3}$$

$$2\nabla\phi \cdot \nabla A_0 + (\Delta\phi)A_0 = 0, \tag{4}$$

$$2\nabla\phi \cdot \nabla A_j + (\Delta\phi)A_j = -\Delta A_{j-1},\ j = 1,2,... \tag{5}$$

The equation for $\phi(r,z)$ is known as the *eikonal* equation while the equations for A_j are known as the *transport* equations. The eikonal equation is solved by introducing a family of curves (rays) which are defined by being perpendicular to the level curves (wavefronts) of $\phi(r,z)$. One finds that the rays satisfy:

$$\frac{dr}{ds} = c\rho(s),\ \frac{d\rho}{ds} = -\frac{1}{c^2}\frac{dc}{dr}, \tag{6}$$

$$\frac{dz}{ds} = c\zeta(s),\ \frac{d\zeta}{ds} = -\frac{1}{c^2}\frac{dc}{dz}, \tag{7}$$

where $(r(s), z(s))$ is the trajectory of the ray in the range-depth plane and $(\rho(s), \zeta(s))$ is the local tangent vector to the ray. Along such a ray, the phase function $\phi(r,z)$ is given by a simple integral:

$$\phi(s) = \int_0^s \frac{c_0}{c(s')} ds'. \tag{8}$$

In addition, it turns out that $A_0(r,z)$ also satisfies a simple differential equation along the ray. In essence, this equation states that the amplitude decays in proportion to the cross-section of a ray tube. Higher order terms in the sequence $A_j(r,z)$ are generally not calculated in ray codes. Neglecting these terms is justified when the frequency is sufficiently high, since for large k the remaining terms $A_j/(ik)^j$ in Eq. (2) go to zero. Unfortunately, it is difficult to predict *a priori* what constitutes a 'sufficiently high' frequency. (A more complete discussion of ray theory may be found in Ref. (1).)

As a simple example, we consider a deep-water scenario with a source located at depth $z_s = 100$ m. The particular sound speed profile and corresponding ray trace are shown in Fig. 1. The rays are obtained by solving Eqns. 7 using a simple integrator.

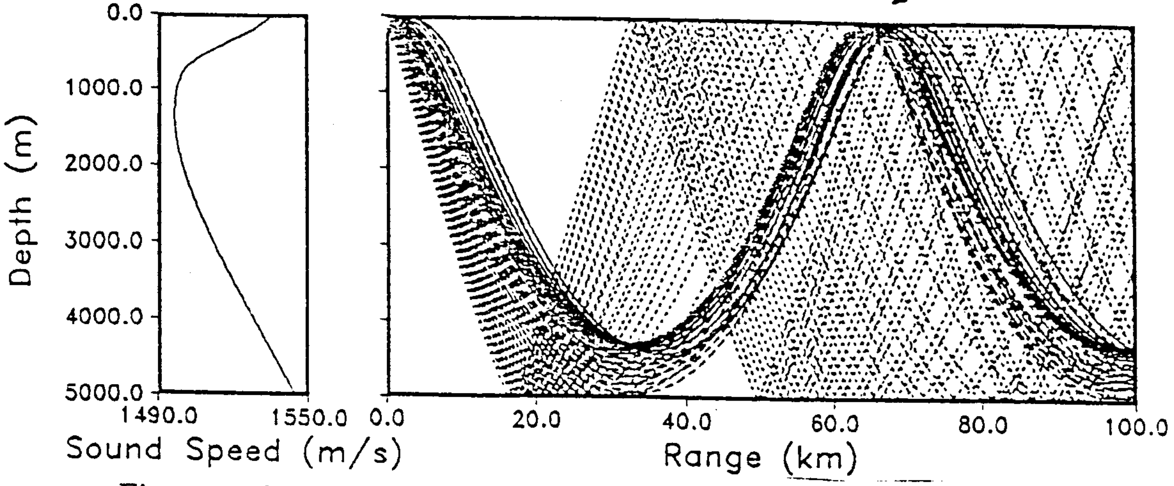

Figure 1: Sound speed profile and ray trace for a deep-water problem

For clarity the different classes of ray paths have been plotted using solid (refracted-refracted), dashed (surface or bottom bounce) and dotted (surface and bottom bounce) lines. The refracted-refracted rays generate the familiar convergence zone pattern with energy cycling up and down the channel.

FAST FIELD PROGRAM

The assumption that the environment is range-independent or stratified leads to a great simplification of the governing equations. In essence the dimension of the problem is reduced by one. Starting with the Helmholtz equation given in Eq. (1), one applies a Fourier-Bessel transform:

$$\hat{p}(k,z) = \int_0^\infty p(r,z) J_0(kr) r\,dr, \tag{9}$$

which leads to:

$$\begin{aligned} \frac{d^2\hat{p}}{dz^2} + \left(\frac{\omega^2}{c^2(z)} - k^2\right)\hat{p} &= \delta(z - z_s), \\ \hat{p}(0) = 0,\ \frac{d\hat{p}}{dz}(D) &= 0, \end{aligned} \tag{10}$$

where for simplicity we have used a pressure release surface boundary condition and a perfectly rigid bottom boundary condition at the depth $z = D$. The solution of this boundary value problem yields $\hat{p}(k,z)$ and the final pressure is then computed using the inverse Fourier-Bessel transform as

$$p(r,z) = \int_0^\infty \hat{p}(k,z) J_0(kr) k\,dk. \tag{11}$$

This is the so-called spectral integral representation of the solution.

One can show that the kernel, $\hat{p}(k,z)$ decays rapidly to zero for $k > K$ where $K = max(\omega/c(z))$. Thus, the integral need only be performed over the truncated interval $[0, K]$. However, the calculation of $p(r,z)$ requires the evaluation of this integral for every single point in range and depth of interest. Fortunately, a trick exists for performing these integrals efficiently. One uses the asymptotic approximation to the Bessel function to obtain,

$$p(r,z) \approx \frac{e^{i\pi/4}}{\sqrt{2\pi r}} \int_0^K \hat{p}(k,z)\sqrt{k} e^{ikr} dk. \tag{12}$$

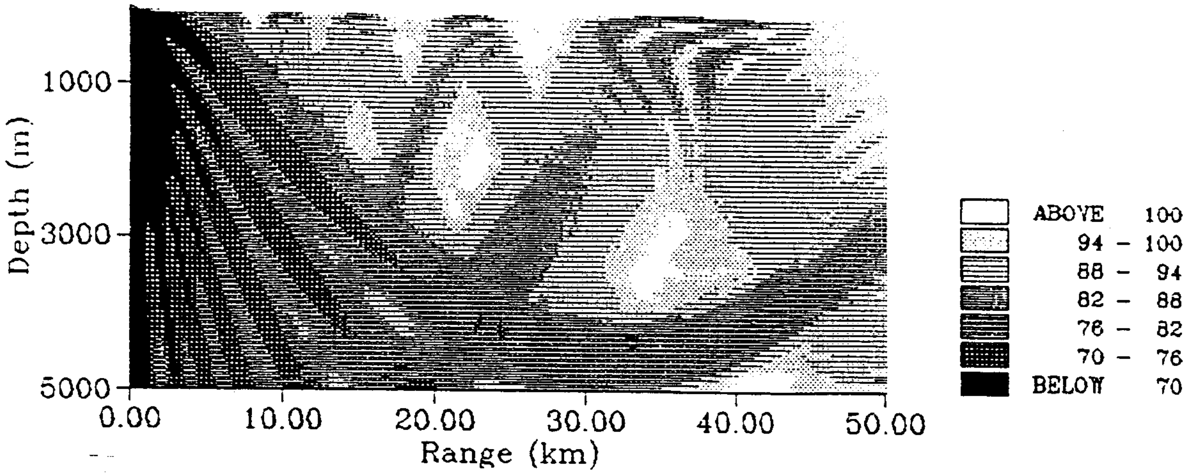

Figure 2: FFP transmission loss for the deep-water problem (50 Hz source)

This has the form of a Fourier transform and can be efficiently evaluated by an FFT.

In summary, the procedure is to solve Eq. (10) for a number of equally spaced k-values to obtain a Green's function, $\hat{p}(k,z)$. This discretely sampled Green's function is then transformed using an FFT to obtain the acoustic pressure versus range.

Methods based on the spectral integral representation have been around for many years. The FFP approach, distinguished by its use of the FFT to calculate the integral was originally suggested by Marsh and extended by DiNapoli (2) in the early 70's. Today there are several implementations of FFP codes with numerous extensions over the 'bare-bones' model described above (see for instance, Ref. (3)). This includes the capability of handling elastic media, interfacial roughness and impulsive sources.

Returning to our previous deep water example, we display the transmission loss plots obtained by an FFP code in Fig. 2. (The transmission loss is $-20\log_{10}(4\pi|p(r,z)|)$). Note the Lloyd mirror pattern which emerges in the near field due to the interference of surface image and direct ray paths. In contrast to the ray solution, the FFP code yields a result which is essentially exact. Starting from the Helmholtz equation for a stratified medium, the only additional approximation is that of using the asymptotic approximation to the Bessel function. This approximation turns out to induce negligible errors beyond a wavelength or so from the source.

NORMAL MODES

Normal mode methods have been widely used for years in underwater acoustics. An early reference, which is widely cited is due to Pekeris (4) who developed the theory for a simple two-layer model of the ocean. Today there are literally dozens of normal mode codes which allow the ocean sound speed profile to be included and in some cases viscoelastic effects in the ocean sub-bottom.

The derivation of the governing equations is straightforward. Again one begins with the Helmholtz equation and seeks a solution as a sum of normal modes:

$$p(r,z) = \sum_{j=1}^{\infty} u_j(z) R_j(r). \tag{13}$$

Substituting the above form into the Helmholtz equation one obtains:

$$\begin{aligned} \frac{d^2 u_j}{dz^2} + \left(\frac{\omega^2}{c^2(z)} - k_j^2\right) u_j &= 0, \\ u_j(0) = 0,\ \frac{du_j}{dz}(D) &= 0, \end{aligned} \tag{14}$$

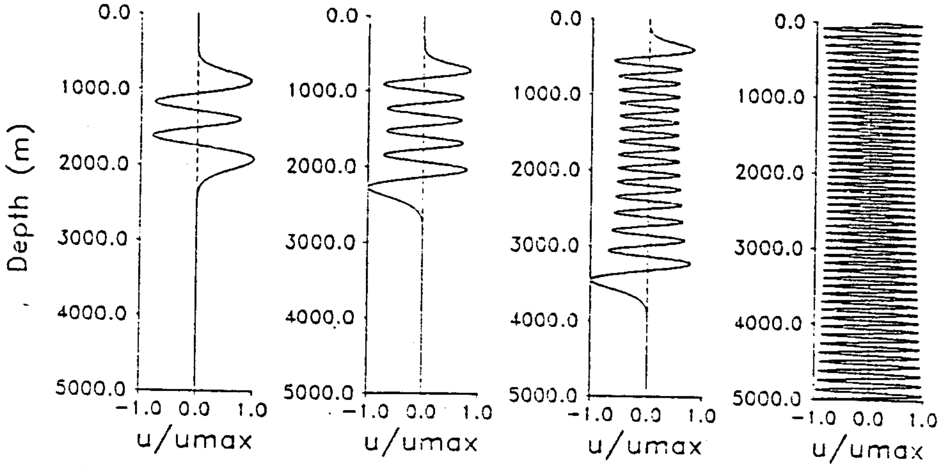

Figure 3: Plots of selected modes for the deep-water problem

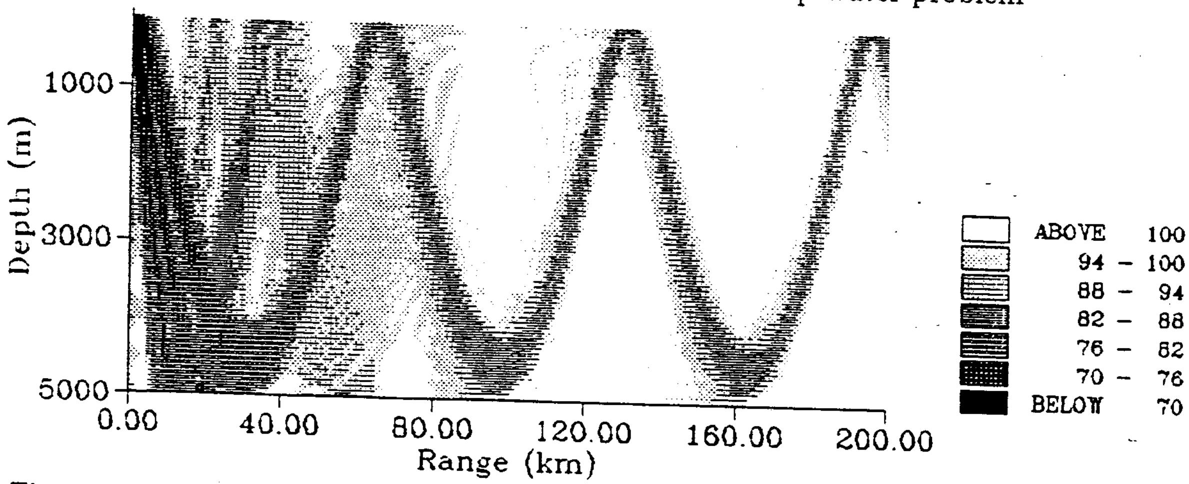

Figure 4: Normal mode transmission loss for the deep-water problem (50 Hz source)

which is identical to the FFP equation of Eq. (10) apart from lacking the forcing term on the right hand side. The above equation has an infinite number of solutions which are like modes of a vibrating string. The modes are characterized by a mode shape function $u_j(z)$ and a horizontal propagation constant k_j (analogous to a frequency of vibration).

The range functions, $R_j(r)$ are found to satisfy,

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial R_j}{\partial r}) + k_j^2 R_j = -u_j(z_s)\frac{\delta(r)}{r}, \tag{15}$$

which has the solution $R_j(r) = \frac{i}{4}u_j(z_s)H_0^{(1)}(k_j r)$. Putting this all together, one finds that,

$$p(r,z) = \frac{i}{4}\sum_{j=1}^{\infty} u_j(z_s)u_j(z)H_0^{(1)}(k_j r), \tag{16}$$

or, using the asymptotic approximation to the Hankel function,

$$p(r,z) \approx \frac{i}{\sqrt{8\pi r}}e^{-i\pi/4}\sum_{j=1}^{\infty} u_j(z_s)u_j(z)\frac{e^{ik_j r}}{\sqrt{k_j}}. \tag{17}$$

In order to apply the method to the deep-water problem, one begins by solving for the modes of Eq. (14) which can be done using standard finite difference techniques. Plots of selected modes are shown in Fig. 3. In general, there are an infinite number of modes and the mode sum must be truncated. For this calculation we include some 200 modes.

The next step is to perform the sum indicated by Eq. (17) to obtain the transmission loss. This result is shown in Fig. 4. The normal modes combine constructively and destructively to reproduce the pattern of convergence zones separated by shadow zones.

The normal mode result is essentially exact beyond the first 50 km or so, a figure which depends on the number of modes that are included in the solution. In the near-field the FFP solution provides a more precise result since the normal mode series is truncated and neglects steep angle ray paths. The advantage of the normal mode approach is that the far-field solution can be calculated very efficiently. Basically, the cost of an FFP solution increases in proportion to range due to the need of sampling the kernel more finely for greater ranges. The opposite is true for a normal mode code: in the near-field more and more modes must be computed. Beyond some intermediate range, say typically 10 water depths, the normal mode solution becomes more efficient.

PARABOLIC EQUATION MODELING

The parabolic equation was introduced in underwater acoustics in 1973 by Tappert and Hardin. One begins by seeking a solution of the Helmholtz equation in the form.

$$p(r,z) = u(r,z)H_0^{(1)}(k_0 r), \tag{18}$$

where k_0 is a reference wavenumber. Substituting into Eq. (1) one finds,

$$u_{rr} + 2ik_0 u_r + k_0^2(n^2 - 1)u + u_{zz} = 0. \tag{19}$$

At this point, one discards the first term to obtain the parabolic equation:

$$u_r = ik_0 \frac{n^2 - 1}{2} u + \frac{i}{2k_0} u_{zz}. \tag{20}$$

This latter step is justified assuming weak range-dependence and narrow angle propagation, i.e. when the dominant energy comes from rays propagating nearly horizontally.

The advantage of the parabolic equation over the original Helmholtz equation is that the PE can be solved by a straightforward marching in range which requires much less computational effort. From a numerical point of view, this range marching is typically implemented using either standard finite difference techniques (5) or using a fast Fourier transform as in the so-called 'split-step' method (6).

In problems with strong range-dependence the PE method is generally the method of choice. The PE method however has several approximations. These approximations introduce errors which increase as the corresponding ray angle increases so that the PE is a narrow angle solution. A great deal of work has been done in the last 10 years or so to construct higher-angle PE's. This increased accuracy generally comes at the expense of computer time.

As an example of the capabilities of PE models we modify our deep-water problem by injecting an idealized seamount into the problem. Referring to the plot in Fig. 5 we observe that the convergence zone path essentially bounces off the seamount and is displaced in range and distorted as a consequence of this interaction. Range-dependent extensions of some of the other models (range-dependent ray theory, adiabatic or coupled modes, coupled FFP) would also be capable treating this problem however at least for the research environment the PE provides the best compromise between accuracy and efficiency for such a problem.

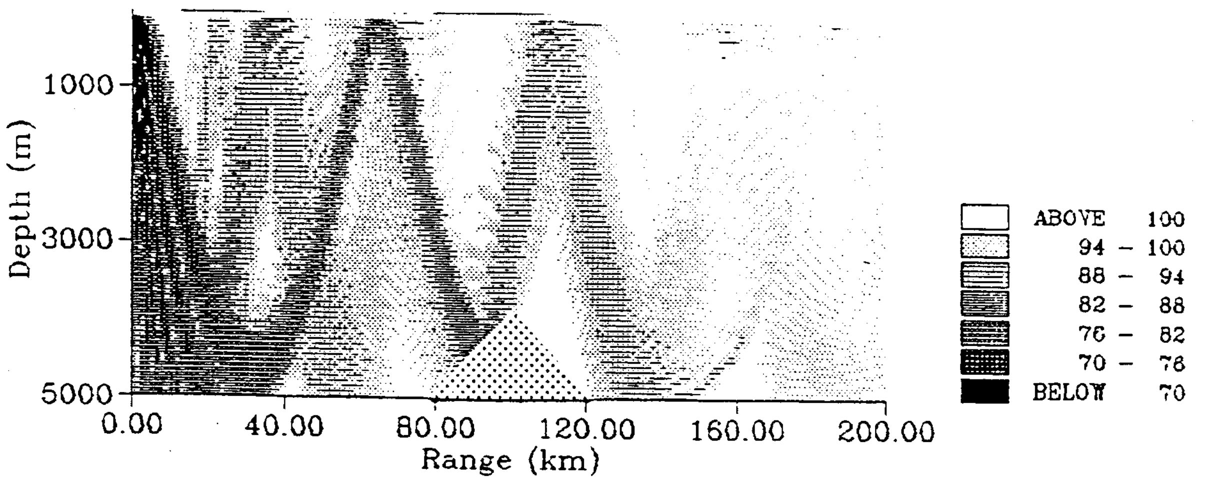

Figure 5: PE transmission loss for the deep-water problem (50 Hz source)

CONCLUSIONS

In the space of so short a paper, it is difficult to do justice to the various models. We mention in passing, that most of these models have been extended to elastic, range-dependent problems and to treat broadband or transient source functions. In addition, full 3D models are emerging largely as a consequence of increased computer power. For a more complete survey we refer the reader to Ref. (7).

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