The Bending of Bonded Layers Due to Thermal Stress

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When two layers expand unequally, but are bonded together, there is a natural tendency for the composite to bend. In this report this problem is addressed in two parts. In the first part we consider the bending of two layers which are bonded together such that there is no slip at the interface. In the second part we consider the same problem when the bonding material allows movement at the interface. The following references are used in this work [1],[2],[3] and [4].

I Bending of two bonded layers with no slip at the interface

In this analysis it is assumed that the two layers behave like beams capable of axial and bending deformations, and there is no slip at the interface. A typical configuration is shown below where, t, E, and α are the thickness, Young's modulus and the coefficient of thermal

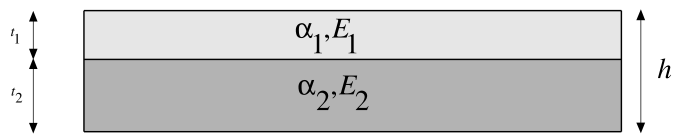

Figure 1: Two bonded layers with different elastic properties.

expansion, respectively. If $\alpha_2 > \alpha_1$, an increase in temperature causes the bottom layer to stretch more than the top layer. Because the two layers can not move with respect to each other at the interface, the whole structure will bend as shown in Fig.(2) The equilibrium of forces yields

$$P_1 = P_2 \tag{1}$$

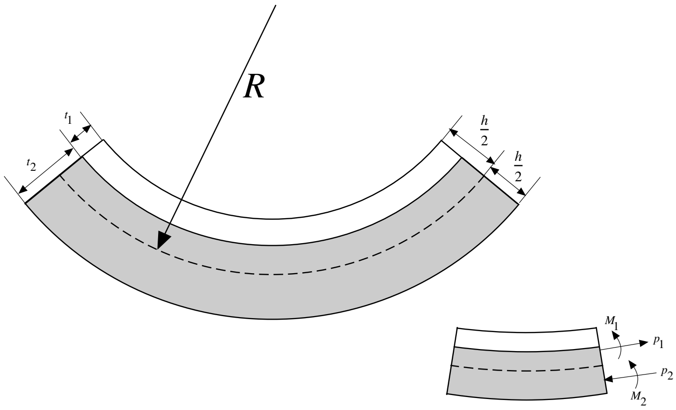

Figure 2: Bonded layers with different elastic properties tend to bend due different responses to temperature.

let $P_1 = P_2 = P$. The equilibrium of moments yields

$$P\frac{h}{2} = M_1 + M_2 \tag{2}$$

Referring to Fig.(3), the strain due to bending is

$$\gamma = \frac{\Delta l}{l} = \frac{l + \Delta l - l}{l} = \frac{2\pi(R + \Delta R) - 2\pi R}{2\pi R} = \frac{\Delta R}{R} = \frac{t}{2R}.$$

The bending moment ,on the other hand is given

$$M = \int_{-t/2}^{t/2} y dF, \tag{3}$$

according to the stress-strain relationship

$$\Delta F = E\frac{\Delta l}{l}\Delta A = \frac{Et}{2R}\Delta A$$

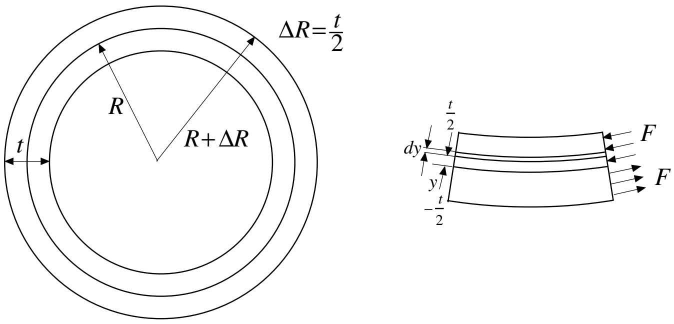

Figure 3:

where A is the cross-sectional area of the bent beam. Using this relationship in Eq.(3), we get

$$M = \frac{E}{R}\int_{-t/2}^{t/2} y^2 dA = \frac{EI}{R}$$

where

$$I = \int_{-t/2}^{t/2} y^2 dA$$

is the moment of inertia of a slice with unit mass per unit area. In our case the cross-section is a rectangle and I becomes

$$I = \int_{-t/2}^{t/2} wy^2 dy = \frac{wt^3}{12}$$

In view the above relationship, Eq.(2) can be written

$$P\frac{h}{2} = M_1 + M_2 = \frac{E_1 I_1 + E_2 I_2}{R} \tag{4}$$

Thermal expansion induces internal tensile and compressive forces and bending. The strain due to these effects for each layer is given

$$\begin{aligned} \gamma_1 &= \alpha_1 T + \frac{P_1}{wt_1 E_1} + \frac{t_1}{2R} \\ \gamma_2 &= \alpha_2 T - \frac{P_2}{wt_2 E_2} - \frac{t_2}{2R}. \end{aligned} \tag{5}$$

Since there is no slipping, $\gamma_1 = \gamma_2$ or

$$\alpha_1 T + \frac{P_1}{wt_1 E_1} + \frac{t_1}{2R} = \alpha_2 T - \frac{P_2}{wt_2 E_2} - \frac{t_2}{2R} \tag{6}$$

from Eq.(4)

$$P_1 = P_2 = P = \frac{2}{h}\left(\frac{E_1 I_1 + E_2 I_2}{R}\right)$$

substituting this into Eq.(6) and solving for $\frac{1}{R}$ gives us the curvature of the bent structure

$$\Gamma = \frac{1}{R} = \frac{(\alpha_2 - \alpha_1)T}{\frac{h}{2} + \frac{2(E_1 I_1 + E_2 I_2)}{hw}(\frac{1}{t_1 E_1} + \frac{1}{t_2 E_2})} \tag{7}$$

where in the above T is the temperature and

$$I_i = \frac{wt_i^3}{12} \qquad i = 1, 2$$

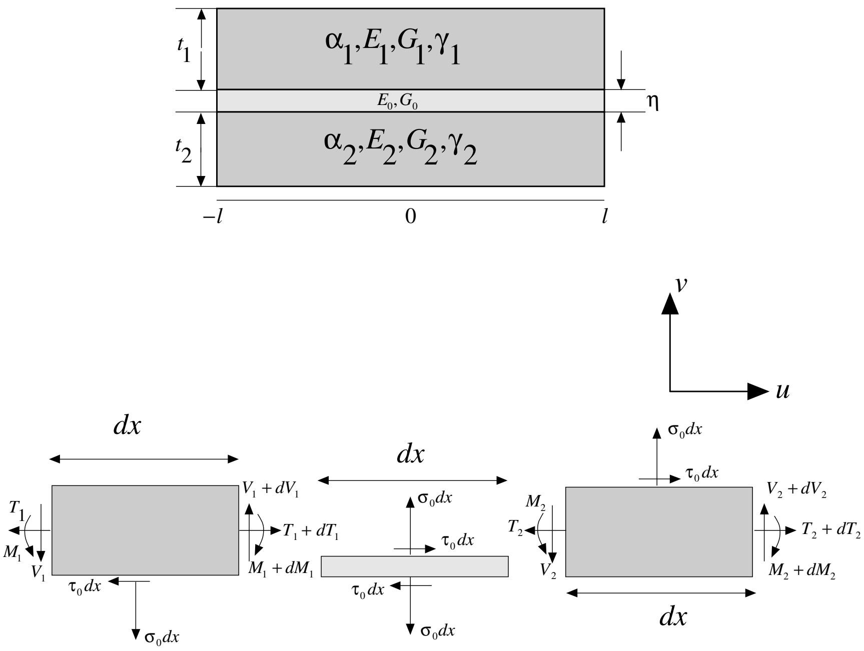

Figure 4:

II Bending of two bonded layers with movement at the interface

Two layers of lengths $2l$, thicknesses of t_1, t_2, and unit widths bonded by an adhesive of thickness η are shown below where γ_i, and G_i, i=1,2 are Poisson's ratio and shear modulus respectively. The other parameters have been defined in Section I. Force and moment diagram of a small section of the above structure is shown below Since the whole structure is in equilibrium, the equilibrium of moments requires that

$$\begin{aligned} dM_1/dx - V_1 + \tau_0 t_1/2 &= 0, \\ dM_2/dx - V_2 + \tau_0 t_2/2 &= 0, \end{aligned} \tag{8}$$

the equilibrium of horizontal forces requires that

$$\begin{aligned} dT_1/dx - \tau_0 &= 0, \\ dT_2/dx + \tau_0 &= 0, \end{aligned} \tag{9}$$

and the equilibrium of vertical forces requires that

$$\begin{aligned} dV_1/dx - \sigma_0 &= 0, \\ dV_2/dx + \sigma_0 &= 0. \end{aligned} \tag{10}$$

From elementary bending theories (see Appendix A) we have

$$\begin{aligned} d^2v_1/dx^2 &= -M_1/D_1 \\ d^2v_2/dx^2 &= -M_2/D_2 \end{aligned} \tag{11}$$

where

$$D_i = \frac{E_i t_i^3}{12(1-\gamma_i^2)}, \qquad i = 1, 2$$

are the flexural rigidities. As we found in Section I, the unit enlongation due to thermal stress and bending are given

$$\begin{aligned} \frac{du_1}{dx} &= \frac{(1-\gamma_1^2)T_1}{E_1 t_1} - \frac{6M_1(1-\gamma_1^2)}{E_1 t_1^2} + (1+\gamma_1)\alpha_1 T \\ \frac{du_2}{dx} &= \frac{(1-\gamma_2^2)T_2}{E_2 t_2} + \frac{6M_2(1-\gamma_2^2)}{E_2 t_2^2} + (1+\gamma_2)\alpha_2 T \end{aligned} \tag{12}$$

Finally, the stress in the joint material is assumed to depend on the displacements (u_1, v_1) and (u_2, v_2) according to the equations

$$\begin{aligned} \tau_0/G_0 &= (u_1 - u_2)/\eta \\ \sigma_0/E_0 &= (v_1 - v_2)/\eta \end{aligned} \tag{13}$$

here G_0 and E_0 are the shear and Young's modulus of the joint material.

Now with the above equations the problem is fully formulated. With appropriate boundary conditions, the analysis is complete. The boundary conditions at $x = l$ are given

$$\begin{aligned} M_1 = M_2 &= 0 \\ T_1 = T_2 &= 0 \\ V_1 = V_2 &= 0 \end{aligned} \tag{14}$$

The above set of equations (Eq.(8) through Eq.(13)) can be reduced to a single sixth-order differential equation for σ_0. A solution of the differential equation can be found containing six constants of integration permitting the six boundary conditions to be satisfied [3]. Following the analysis in [3], the differential equation for σ_0 is given

$$\frac{d^6\sigma_0}{dx^6} - \frac{G_0 c}{\eta}\frac{d^4\sigma_0}{dx^4} + \frac{E_0 b}{\eta}\frac{d^2\sigma_0}{r dx^2} - \frac{G_0 E_0 (bc - a^2)\sigma_0}{\eta^2} = 0 \tag{15}$$

where the constants a, b, and c are defined as

$$a = 6\left[\frac{(1-\gamma_1^2)}{E_1 t_1^2} - \frac{(1-\gamma_2^2)}{E_2 t_2^2}\right],$$

$$b = 12\left[\frac{(1-\gamma_1^2)}{E_1 t_1^3} + \frac{(1-\gamma_2^2)}{E_2 t_2^3}\right],$$

$$c = 4\left[\frac{(1-\gamma_1^2)}{E_1 t_1} + \frac{(1-\gamma_2^2)}{E_2 t_2}\right].$$

The solution of Eq.(15) is related to the roots of the algebraic equation

$$y^3 - \frac{G_0 c}{\eta} y^2 + \frac{E_0 b}{\eta} y - \frac{G_0 E_0 (bc - a^2)}{\eta^2} = 0$$

let

$$\begin{aligned} a_0 &= -\frac{G_0 E_0 (bc - a^2)}{\eta^2}, \\ a_1 &= \frac{E_0 b}{\eta}, \\ a_2 &= -\frac{G_0 c}{\eta}, \\ r &= \frac{(a_1 a_2 - 3a_0)}{6} - \frac{a_2^3}{27}, \\ q &= \frac{a_1}{3} - \frac{a_2^2}{9}. \end{aligned}$$

then the roots of the above algebraic equation are

$$\begin{aligned} y_1 &= \beta_1, \\ y_2 &= \beta_H + i\beta_V, \\ y_3 &= \beta_H - i\beta_V. \end{aligned}$$

where

$$\begin{aligned} \beta_1 &= (s_1 + s_2) - \frac{a_2}{3}, \\ \beta_H &= -\frac{1}{2}(s_1 + s_2) - \frac{a_2}{3}, \\ \beta_V &= \frac{\sqrt{3}}{2}(s_1 - s_2). \end{aligned}$$

and

$$s_1 = \sqrt{r + \sqrt{q^3 + r^2}}$$

$$s_2 = \sqrt{r - \sqrt{q^3 + r^2}}$$

then the solution to (Eq.15) is given

$$\sigma_0 = A_1 \cosh \beta_1 x + A_3 \cosh \beta_H x \cos \beta_V x + A_5 \sinh \beta_H \sin \beta_V x. \tag{16}$$

the constants A_1, A_3, and A_5 are determined by the boundary conditions (Eq.14). Similarly the shear stress τ_0 is determined to be

$$\tau_0 = C_1 \sinh \beta_1 x + C_2 \sinh \beta_H x \cos \beta_V x + C_3 \cosh \beta_H x \sin \beta_V x. \quad (17)$$

The details of expressions for the constants $A_1..., C_3$ are given in Appendix B.

The radius of curvature of a bent layer, at least to our beam approximation, is proportional to the bending moments M_1 and M_2. M_1 and M_2 can be found from the expressions for σ_0 and τ_0 by integrating the set of equations (Eq.8) and (Eq.10). The constants of integration can be found from the boundary conditions (Eq.14). Since the integration is straight forward, we skip the details and write down the functional form of $M_1(x)$ and $M_2(x)$. All the constants appearing in these expressions are given in Appendix B.

$$M_1(x) = \psi_1 \cosh \beta_1 x + \xi_1 \sinh \beta_H x \sin \beta_V x + \xi_2 \cosh \beta_H x \cos \beta_V x + \Omega_1 x + \Theta_1, \quad (18)$$

and

$$M_2(x) = \zeta_1 \cosh \beta_1 x - \xi_3 \sinh \beta_H x \sin \beta_V x + \xi_4 \cosh \beta_H x \cos \beta_V x + \Omega_2 x + \Theta_2. \quad (19)$$

III Appendix A

We found in the main text that the unit enlongation in x and y directions of an element *abcd* at distance z from the neutral surface are

$$\epsilon_x = \frac{z}{\rho_x}, \qquad \epsilon_y = \frac{z}{\rho_y}$$

where, ρ_x and ρ_y are the radii of curvature in the x and y directions. From Hook's law

$$\epsilon_x = \frac{1}{E}(\sigma_x - \gamma \sigma_y), \quad \text{(A-1)}$$

$$\epsilon_y = \frac{1}{E}(\sigma_y - \gamma \sigma_x). \quad \text{(A-2)}$$

where γ is Poisson's ratio. From the above equations we find

$$\sigma_x = \frac{Ez}{1-\gamma^2}(\frac{1}{\rho_x} + \gamma \frac{1}{\rho_y}), \quad \text{(A-3)}$$

and

$$\sigma_y = \frac{Ez}{1-\gamma^2}(\frac{1}{\rho_y} + \gamma \frac{1}{\rho_x}). \quad \text{(A-4)}$$

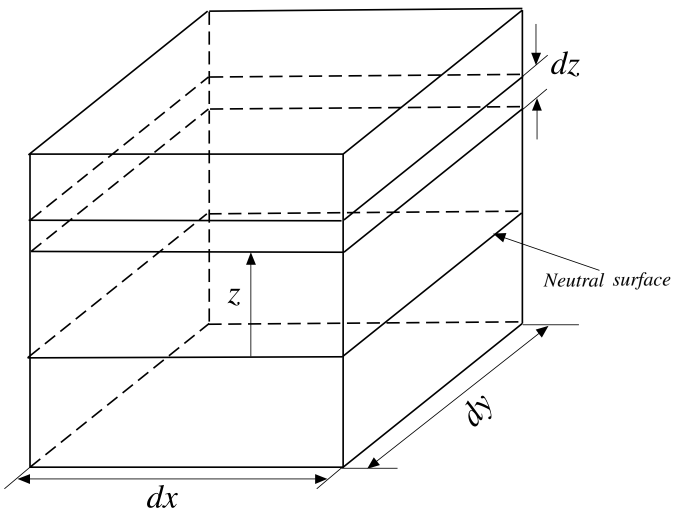

Figure 5:

The normal stress distribution over the lateral sides of the element in the above figure can be reduced to couples which must be equal to the bending moments

$$\int_{-h/2}^{h/2} \sigma_x z dz dy = M_x dy,$$

$$\int_{-h/2}^{h/2} \sigma_y z dz dx = M_y dx.$$

Substituting from (Eq.A-3) and (Eq.A-4) for σ_x and σ_y gives

$$M_x = D(\frac{1}{\rho_x} + \gamma \frac{1}{\rho_y}),$$

and

$$M_y = D(\frac{1}{\rho_y} + \gamma \frac{1}{\rho_x}).$$

where

$$D = \frac{E}{1-\gamma^2} \int_{-h/2}^{h/2} z^2 dz = \frac{Eh^3}{12(1-\gamma^2)}$$

is the flexural rigidity of the plate. In our analysis in the main text we have assumed that the bending occurs only in one direction which means that the relation between the bending moment and flexural rigidity reduces to

$$M = \frac{D}{\rho} \tag{A-5}$$

From differential geometry, on the other hand, we know that the curvature of a bent beam is given

$$\frac{1}{\rho} = -\frac{\frac{d^2v}{dx^2}}{(1+(\frac{dv}{dx})^2)^{3/2}}$$

where v denotes the deflection of the beam. For small deflections this relation reduces to

$$\frac{1}{\rho} \approx -\frac{d^2v}{dx^2}$$

From (Eq.A-5) we find

$$\frac{d^2v}{dx^2} = -\frac{M}{D}.$$

IV Appendix B

In order to be able to write down more compact expressions for the constants appearing in the main text, we define the following parameters:

$$\tau = \frac{(9E_0G_0)}{\eta^2}[(1+\gamma_1)\alpha_1 - (1+\gamma_2)\alpha_2)]T$$

$$\begin{aligned}
CHC &= \cosh\beta_H l \cos\beta_V l,\\
SHS &= \sinh\beta_H l \sin\beta_V l,\\
CHS &= \cosh\beta_H l \sin\beta_V l,\\
SHC &= \sinh\beta_H l \cos\beta_V l,\\
\beta_s &= (\beta_H^2 + \beta_V^2),\\
\beta_d &= (\beta_H^2 - \beta_V^2),\\
\beta_m &= \beta_H\beta_V,\\
p_1 &= \beta_1^2 \cosh\beta_1 l,\\
p_2 &= \beta_d CHC - 2\beta_m SHS,\\
p_3 &= \beta_d SHS + 2\beta_m CHC,\\
p_4 &= \frac{\sinh\beta_1 l}{\beta_1},\\
p_5 &= \frac{\beta_V}{\beta_s}CHS + \frac{\beta_H}{\beta_s}SHC,
\end{aligned}$$

$$
\begin{aligned}
p_6 &= \frac{\beta_H}{\beta_s} CHS - \frac{\beta_V}{\beta_s} SHC, \\
p_7 &= (\beta_1^4 + \frac{E_0 b}{\eta}) \cosh \beta_1 l, \\
p_8 &= [\beta_d - 4\beta_m^2 + \frac{E_0 b}{\eta}] CHC - 4\beta_m \beta_d SHS, \\
p_9 &= [\beta_d - 4\beta_m^2 + \frac{E_0 b}{\eta}] SHS + 4\beta_m \beta_d CHC.
\end{aligned}
$$

The constants A_1, A_3, and A_5 are then given

$$A_1 = \frac{\tau(p_2 p_6 - p_3 p_5)}{Den},$$

$$A_3 = \frac{\tau(p_3 p_4 - p_1 p_6)}{Den},$$

$$A_5 = \frac{\tau(p_1 p_5 - p_2 p_4)}{Den},$$

where

$$Den = p_7(p_2 p_6 - p_3 p_5) + p_8(p_3 p_4 - p_1 p_6) + p_9(p_1 p_5 - p_2 p_4)$$

from here C_1, C_2, and C_3 are found to be

$$
\begin{aligned}
C_1 &= \frac{\eta}{\beta_1 E_0 a} (\beta_1^4 + \frac{E_0 b}{\eta}) A_1, \\
C_2 &= \frac{\eta}{E_0 a} (\gamma_1 A_3 - \gamma_2 A_5), \\
C_3 &= \frac{\eta}{E_0 a} (\gamma_1 A_5 + \gamma_2 A_3).
\end{aligned}
$$

Next we define the following set of parameters

$$
\begin{aligned}
\phi_1 &= \frac{A_1}{\beta_1}, \\
\phi_2 &= \frac{(A_3 \beta_V + A_5 \beta_H)}{\beta_s}, \\
\phi_3 &= \frac{(A_3 \beta_H - A_5 \beta_V)}{\beta_s}, \\
\psi_1 &= \frac{(\phi_1 - C_1 \frac{t_1}{2})}{\beta_1},
\end{aligned}
$$

$$
\begin{aligned}
\psi_2 &= \frac{(\phi_2 - C_3\frac{t_1}{2})}{\beta_s}, \\
\psi_3 &= \frac{(\phi_3 - C_2\frac{t_1}{2})}{\beta_s}, \\
\zeta_1 &= \frac{(\phi_1 + C_1\frac{t_2}{2})}{\beta_1}, \\
\zeta_2 &= \frac{(\phi_2 + C_3\frac{t_2}{2})}{\beta_s}, \\
\zeta_3 &= \frac{(\phi_3 + C_2\frac{t_2}{2})}{\beta_s}, \\
\Omega_1 &= -(\phi_1 \sinh \beta_1 l + \phi_2 CSH + \phi_3 SHC), \\
\Omega_2 &= -\Omega_1, \\
\xi_1 &= \psi_2\beta_H + \psi_3\beta_V, \\
\xi_2 &= \psi_3\beta_H - \psi_2\beta_V, \\
\xi_3 &= \zeta_2\beta_H + \zeta_3\beta_V, \\
\xi_4 &= \zeta_2\beta_V - \zeta_3\beta_H, \\
\Theta_1 &= -\psi_1 \cosh \beta_1 l - \xi_1 SHS - \xi_2 CHC - \Omega_1 l, \\
\Theta_2 &= \zeta_1 \cosh \beta_1 l + \xi_3 SHS - \xi_4 CHC - \Omega_2 l.
\end{aligned}
$$

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