An energy-conserving one-way coupled mode propagation model

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The equations of motion for pressure and displacement fields in a waveguide have been used to derive an energy-conserving, one-way coupled mode propagation model. This model has three important properties: First, since it is based on the equations of motion, rather than the wave equation, instead of two coupling matrices, it only contains one coupling matrix. Second, the resulting coupling matrix is anti-symmetric, which implies that the energy among modes is conserved. Third, the coupling matrix can be computed using the local modes and their depth derivatives. The model has been applied to two range-dependent cases: Propagation in a wedge, where range dependence is due to variations in water depth and propagation through internal waves, where range dependence is due to variations in water sound speed. In both cases the solutions are compared with those obtained from the parabolic equation (PE) method. © *2002 Acoustical Society of America.* [DOI: 10.1121/1.1419088]

I. INTRODUCTION

The coupled mode theory commonly used in acoustics was originally derived by Pierce[1] and Milder[2] from the wave equation for the velocity potential. In this formulation the field in a range-dependent waveguide is expanded in terms local modes with range-dependent coefficients (mode amplitudes). The application of the continuity of pressure and the vertical component of particle velocity allows a partial separation of the depth and range variables and yields a system of second order coupled differential equations for the mode amplitudes. However, as is pointed out by Rutherford and Hawker,[3] while the boundary condition of continuity of vertical component of particle velocity is correct for horizontal boundaries, when applied to problems with nonhorizontal boundaries, this boundary condition is only an approximation to the correct boundary condition of the continuity of the normal component of particle velocity. Rutherford and Hawker showed that one consequence of this approximation is nonconservation of energy. They used the WKBJ method to obtain a solution which satisfies both the proper boundary condition and conserved energy to first order in the slope of the nonhorizontal boundaries and interfaces. This problem was also addressed by Fawcett,[4] who derived a system of coupled mode equations which satisfies the correct boundary conditions. However, in addition to the two coupling matrices, which is typical of all coupled mode theories derived from the wave equation, the equations derived by Fawcett also contain two other so-called interface matrices. These matrices require a knowledge of the range derivatives of the local modes, which can only be computed approximately. The inaccuracy resulting from this along with the complexity involved in solving the system of differential equations make this an impractical computational method for solving range-dependent problems.

In an attempt to reduce the complexity involved in computing the coupling matrices, McDonald[5] used the original Pierce–Milder equations to argue that for a waveguide whose horizontal length scales are much larger than the acoustic wavelength only one of the two coupling matrices has significant contribution. By neglecting one of the coupling matrices and the horizontal derivative of the density, McDonald was able to derive an expression for the remaining coupling matrix in terms of local modes and their depth derivatives. The expression for the coupling matrix derived by McDonald was used by Abawi *et al.*[6] who derived a system of one-way coupled mode equations for the mode amplitudes. Although this method is a practical computational method for solving range-dependent problems, it still suffers from approximations made by neglecting one of the two coupling matrices. More importantly, since this method is based on the same boundary conditions as those used by Pierce and Milder, the energy among modes is not conserved.

The coupled mode model that is presented in this paper is derived, not from the wave equation, as is the case for the Pierce–Milder method, but from the equations of motion for the pressure and displacement fields. This method, which was first used by Shevchenko,[7] has common use in seismology and geophysics, Odom,[8] Maupin[9] and Tromp.[10] The derivation in this paper follows the derivation of Tromp. While the model derived by Tromp is for the general elastic waveguide, this model is derived for a waveguide consisting of fluid layers. Since the equations of motion constitute a system of two first order coupled differential equations, the coupled mode equations resulting from them only contain a single coupling matrix. Furthermore, this method provides a natural framework for applying the correct boundary and interface conditions without adding any more complexity to the numerical solution of the equations. In fact, it is shown in this paper that the proper application of the boundary and interface conditions not only simplifies the numerical computation of the coupling matrix by allowing it to be expressed in terms the local modes and their depth derivatives, it also makes it possible to show that the resulting coupling matrix is anti-symmetric, which guarantees the conservation of energy among modes.[3,6,10]

The method presented in this paper and the references

cited in the above fall in the category of continuous coupled mode theory where the solution of the wave equation in a range-dependent waveguide is obtained by solving of a set of coupled differential equations. The solution for a range-dependent waveguide can also be obtained by the discrete coupled mode method.[11] In this method the range-dependent waveguide is approximated by range-independent stair steps and the coupled mode solution is obtained by matching the solutions of the wave equation for neighboring stair steps at their common boundary. This method is easy to implement numerically and has wide application in ocean acoustics. However, the method that is presented in this paper has two advantages over the discrete coupled mode method. The first advantage is it expresses the coupling matrix in terms of physical parameters and thus provides insight into the process of mode coupling by clearly showing what is responsible for it. The other, more important advantage is that this method in principle can be extended to handle propagation in three dimensions, where the discrete coupled mode method is designed for propagation in two dimensions and there is no obvious way to modify it to handle propagation in three dimensions.

This paper is organized in the following way. In Sec. II the coupled mode model is derived, where some of the details of the derivation are given in the appendices. In Sec. III the model is applied to two range-dependent propagation scenarios. The first one is propagation in a wedge, where range-dependence is entirely due to variations in water depth. The parameters used in this example are scaled to match those used by Coppens and Sanders[12] in a model tank experiment. The second example is propagation through internal waves, where range-dependence is entirely due to the variations in water sound speed. The parameters used in this example were those used in a test case in the Shallow Water Acoustics Modeling workshop.[13] In both of the above examples the results obtained from the coupled mode model are compared with those obtained from the parabolic equation (PE) method.[14]

II. DERIVATION OF THE COUPLED-MODE EQUATIONS

Consider the equations of motion with the x-axis in the direction of propagation

$$\partial_x p = \rho \omega^2 u_x,$$

$$\partial_x u_x = -\frac{p}{\rho c^2} - \hat{\mathbf{z}} \cdot \partial_z \mathbf{u},$$

$$\hat{\mathbf{z}} \cdot \mathbf{u} = \frac{1}{\rho \omega^2} \partial_z p.$$

In the above equations p is the pressure and $\mathbf{u}$ is the displacement vector. The pressure and the normal component of the displacement are continuous across any interface. This may be expressed as

$$[p]_-^+ = 0, \quad [\hat{\mathbf{n}} \cdot \mathbf{u}]_-^+ = 0,$$

where $[\zeta]_-^+ = \zeta_+ - \zeta_-$, the $+/-$ indicate the value of the parameter ζ just above/below the interface and $\hat{\mathbf{n}}$ is the unit vector normal to the interface.

The field quantities can be expressed as the sum of normal modes

$$p(x,z) = \sum_n p_n(z) e^{ik_n x}, \quad \mathbf{u}(x,z) = \sum_n \mathbf{u}_n(x,z) e^{ik_n x},$$

where p_n and $\mathbf{u}_n$ denote the normal modes. Substituting the above expressions into the equations of motion results in the following relationships for the modes

$$ik_n p_n = \rho \omega^2 u_n,$$

$$ik_n u_n = -\frac{p_n}{\rho c^2} - \partial_z \left(\frac{1}{\rho \omega^2} \partial_z p_n \right),$$

$$\hat{\mathbf{z}} \cdot \mathbf{u}_n = \frac{1}{\rho \omega^2} \partial_z p_n, \tag{1}$$

$$[p_n]_-^+ = 0, \quad [\hat{\mathbf{z}}_n \cdot \mathbf{u}_n]_-^+ = 0.$$

In the above equations u_n denotes the x component of the displacement.

In a range-dependent environment the pressure and the displacement vector may be expressed as a sum of local normal modes with range-dependent coefficients, $c_n(x)$

$$p(x,z) = \sum_n c_n(x) p_n(z;x) e^{ik_n x},$$

$$\mathbf{u}(x,z) = \sum_n c_n(x) \mathbf{u}_n(z;x) e^{ik_n x}. \tag{2}$$

In this notation the parametric range-dependence of the local modes at range x is indicated by the semicolon separating z and x.

Substitution of the above expansion into the equations of motion gives

$$\partial_x \sum_n c_n p_n e^{ik_n x} = \sum_n \rho \omega^2 c_n u_n e^{ik_n x},$$

$$\partial_x \sum_n c_n u_n e^{ik_n x} = \sum_n \left(-\frac{1}{\rho c^2} p_n - \hat{\mathbf{z}} \cdot \partial_z \mathbf{u}_n \right) c_n e^{ik_n x}.$$

Multiplying the first equation by $u_m e^{-ik_m x}$ and the second equation by $p_m e^{-ik_m x}$, adding the two equations and integrating along the depth of the waveguide gives

$$\sum_n \int_0^B \{(u_n p_m + p_n u_m) \partial_x c_n + c_n (p_m \partial_x u_n + u_m \partial_x p_n)$$

$$+ ic_n k_n (u_n p_m + p_n u_m)\} e^{i(k_n - k_m)x} \, dz$$

$$= \sum_n \int_0^B \left\{ -\frac{1}{\rho c^2} p_n p_m - \hat{\mathbf{z}} \cdot (\partial_z \mathbf{u}_n) p_m \right.$$

$$\left. + \rho \omega^2 u_n u_m \right\} c_n e^{i(k_n - k_m)x} \, dz. \tag{3}$$

Since according to Eq. (1) $u_n = ik_n p_n / (\rho \omega^2)$, we find

$$2\,\partial_x(c_m)k_m + c_m\,\partial_x k_m = \sum_{n\neq m} A_{mn}c_n. \tag{4}$$

The details of the above derivation is given in Appendix B. In the above equation, A_{mn} is the coupling matrix given by

$$A_{mn} = -\left[(k_n+k_m)\int_0^B \frac{1}{\rho}p_m\,\partial_x(p_n)dz + k_n\int_0^B p_n p_m\right.$$
$$\left.\times\,\partial_x\!\left(\frac{1}{\rho}\right)dz + \left[\frac{k_n}{\rho}p_n p_m \partial_x h\right]_-^+\right]e^{i(k_n-k_m)x}. \tag{5}$$

The above equation is not yet in the desired form, as it contains the range derivative of the modes, which is difficult to compute accurately. In the remainder of this section we will use the modal equations and the boundary and interface conditions to convert the above equation into one which only contains the local modes and their depth derivatives.

Consider the mode equations for mode n and mode m

$$\partial_z\!\left(\frac{1}{\rho}\partial_z p_n\right) + \frac{1}{\rho}(k^2-k_n^2)p_n = 0, \tag{6}$$

$$\partial_z\!\left(\frac{1}{\rho}\partial_z p_m\right) + \frac{1}{\rho}(k^2-k_m^2)p_m = 0. \tag{7}$$

The modal equation is obtained by substituting $u_n = ik_n p_n/(\rho\omega^2)$ into the second equation in Eq. (1). Next evaluate

$$\int_0^B \{p_m\,\partial_x[\text{Eq. }(6)] - [\text{Eq. }(7)]\partial_x p_n\}dz.$$

This gives

$$\int_0^B \left\{\partial_z\!\left(\partial_x\!\left(\frac{1}{\rho}\right)\partial_z p_n\right)p_m + \partial_z\!\left(\frac{1}{\rho}\partial_z(\partial_x p_n)\right)p_m\right.$$
$$+\frac{1}{\rho}p_n p_m\,\partial_x(k^2) + (k^2-k_n^2)\partial_x\!\left(\frac{1}{\rho}\right)p_n p_m$$
$$\left.+(k_m^2-k_n^2)\frac{1}{\rho}p_m\,\partial_x p_n - \partial_z\!\left(\frac{1}{\rho}\partial_z p_m\right)\partial_x p_n\right\}dz = 0.$$

The fifth term in the above equation can be written as

$$(k_m-k_n)(k_m+k_n)\int_0^B \frac{1}{\rho}p_m\,\partial_x p_n\,dz$$

$$= -\int_0^B\left\{\partial_z\!\left(\partial_x\!\left(\frac{1}{\rho}\right)\partial_z p_n\right)p_m + \partial_z\!\left(\frac{1}{\rho}\partial_z(\partial_x p_n)\right)p_m\right.$$
$$+\frac{1}{\rho}p_n p_m\,\partial_x(k^2) + (k^2-k_n^2)\partial_x\!\left(\frac{1}{\rho}\right)p_n p_m$$
$$\left.-\partial_z\!\left(\frac{1}{\rho}\partial_z p_m\right)\partial_x p_n\right\}dz,$$

or

$$(k_m+k_n)\int_0^B \frac{1}{\rho}p_m\,\partial_x p_n\,dz$$

$$= (k_n-k_m)^{-1}\int_0^B\left\{\partial_z\!\left(\partial_x\!\left(\frac{1}{\rho}\right)\partial_z p_n\right)p_m\right.$$

$$+\partial_z\!\left(\frac{1}{\rho}\partial_z(\partial_x p_n)\right)p_m + \frac{1}{\rho}p_n p_m \partial_x(k^2)$$

$$\left.+(k^2-k_n^2)\partial_x\!\left(\frac{1}{\rho}\right)p_n p_m - \partial_z\!\left(\frac{1}{\rho}\partial_z p_m\right)\partial_x p_n\right\}dz. \tag{8}$$

We would like to transfer all terms involving the range derivatives of the modes from inside the integral to the boundary term by using integration by parts. The first term can be written as

$$\int_0^B \partial_z\!\left(\partial_x\!\left(\frac{1}{\rho}\right)\partial_z p_n\right)p_m\,dz$$

$$= \left[\partial_x\!\left(\frac{1}{\rho}\right)(\partial_z p_n)p_m\right]_-^+ - \int_0^B \partial_x\!\left(\frac{1}{\rho}\right)(\partial_z p_m)(\partial_z p_n)dz.$$

Similarly, the second and the fifth terms can be written as

$$\int_0^B \partial_z\!\left(\frac{1}{\rho}\partial_z(\partial_x p_n)\right)p_m\,dz$$

$$= \left[\frac{1}{\rho}\partial_z(\partial_x p_n)p_m\right]_-^+ - \int_0^B \frac{1}{\rho}\partial_z(p_n)\partial_z(p_m)dz,$$

and

$$-\int_0^B \partial_z\!\left(\frac{1}{\rho}\partial_z p_m\right)\partial_x(p_n)dz$$

$$= -\left[\frac{1}{\rho}\partial_x(p_n)(\partial_z p_m)\right]_-^+ + \int_0^B \frac{1}{\rho}\partial_z(p_n)\partial_z(p_m)dz.$$

Substituting these into Eq. (8) gives

$$(k_m+k_n)\int_0^B \frac{1}{\rho}p_m\,\partial_x p_n dz$$

$$= (k_n-k_m)^{-1}\int_0^B\left\{\left(-\partial_x\!\left(\frac{1}{\rho}\right)(\partial_z p_m)(\partial_z p_n)\right)\right.$$

$$\left.+\frac{1}{\rho}p_n p_m\,\partial_x(k^2) + (k^2-k_n^2)\partial_x\!\left(\frac{1}{\rho}\right)p_n p_m\right\}dz$$

$$+\left[\partial_x\!\left(\frac{1}{\rho}\right)(\partial_z p_n)p_m + \frac{1}{\rho}\partial_x(\partial_z p_n)p_m\right.$$

$$\left.-\frac{1}{\rho}\partial_x(p_n)(\partial_z p_m)\right]_-^+. \tag{9}$$

The boundary term in the above equation may be written as

$$\left[\partial_x\!\left(\frac{1}{\rho}\right)(\partial_z p_n)p_m + \frac{1}{\rho}\partial_x(\partial_z p_n)p_m - \frac{1}{\rho}\partial_x(p_n)(\partial_z p_m)\right]_-^+$$

$$= \left[\partial_x\!\left(\frac{1}{\rho}\partial_z p_n\right)p_m - \partial_x(p_n)\frac{1}{\rho}(\partial_z p_m)\right]_-^+.$$

Since the derivative along the interface of a continuous function f is continuous, we have

$$[\hat{\mathbf{T}} \cdot \nabla f]_-^+ = 0, \quad \text{where} \quad \hat{\mathbf{T}} = \hat{\mathbf{x}} + \frac{\partial h}{\partial x} \hat{\mathbf{z}}.$$

This gives

$$[\partial_x f]_-^+ = -[\partial_x(h)\partial_z(f)]_-^+.$$

Since both p_n and $\partial_z p_n/\rho$ are continuous across the interface $z = h(x)$, the above boundary term can be written as

$$\left[\partial_x(h)\partial_z(p_n)\frac{1}{\rho}(\partial_z p_m) - \partial_x h \ \partial_z\left(\frac{1}{\rho}\partial_z p_n\right)p_m \right]_-^+.$$

With the help of the wave equation, Eq. (6), this may be written as

$$\left[\partial_x(h)\partial_z(p_n)\frac{1}{\rho}(\partial_z p_m) + \partial_x h \frac{1}{\rho}(k^2 - k_n^2)p_n p_m \right]_-^+.$$

Substituting this into Eq. (9) and the result into Eq. (5) yields

$$(k_m - k_n)A_{mn} = \left\{ \int_0^B \left[(k^2 - k_n k_m)p_n p_m \right. \right.$$

$$- \partial_z p_n \ \partial_z p_m]\partial_x\left(\frac{1}{\rho}\right) + \frac{p_n p_m}{\rho}\partial_x(k^2) \right]dz$$

$$+ \partial_x h \left[\frac{1}{\rho}\partial_z p_n \ \partial_z p_m + \frac{1}{\rho}(k^2 \right.$$

$$\left. \left. - k_n k_m)p_n p_m \right]_-^+ \right\} e^{i(k_n - k_m)x}. \tag{10}$$

The expression for the coupling matrix given by the above equation is the main result of this paper. It shows the effect of mode coupling due to contribution from volumetric and bathymetric variations in the waveguide separately. The first part of the coupling matrix containing the integral is due to contribution from volumetric variations in the waveguide such as variations in sound speed and density as a function of range. The second part is due to contribution from bathymetric variations in range, i.e., variations in water depth, as is evident from the presence of $\partial_x h$.

The coupling matrix has two important properties. First, it is anti-symmetric, i.e., $A_{mn} = -A_{nm}^\dagger$. This implies that en-

Coupled Mode Solution

Depth Slice at 30 m

Adiabatic Mode Solution
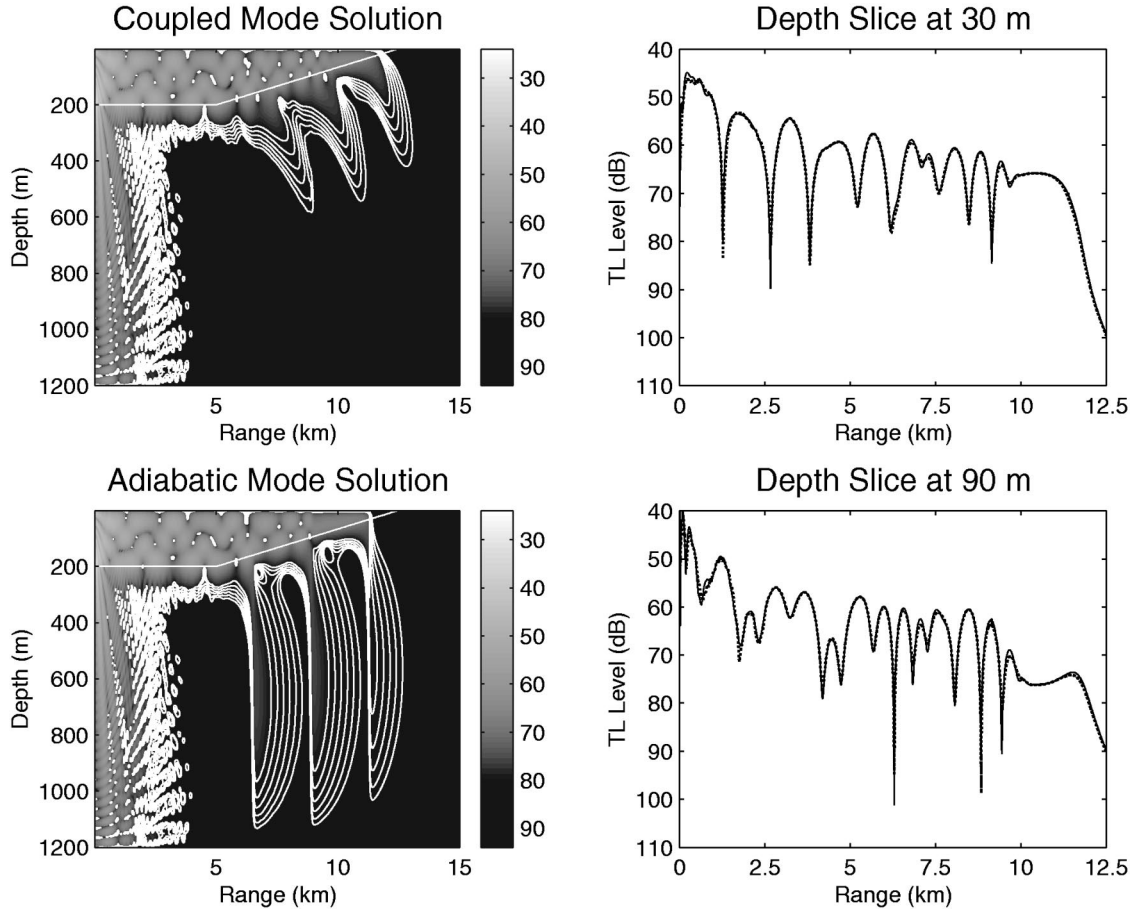
Depth Slice at 90 m

FIG. 1. Propagation in an oceanic wedge: The water depth is constant at 200 m for the first 5 km and it slowly decreases to zero in the next 7.5 km resulting in a wedge angle of 1.55 deg. The top left panel shows the acoustic field due a 25 Hz source placed at 180 m as a function range and depth computed using the coupled mode model. The bottom left panel shows the acoustic field computed using the adiabatic mode model. The top and bottom panels on the right show a comparison of the transmission loss as a function of range computed using the PE model (solid) the coupled mode model (dotted) for two receiver depths.
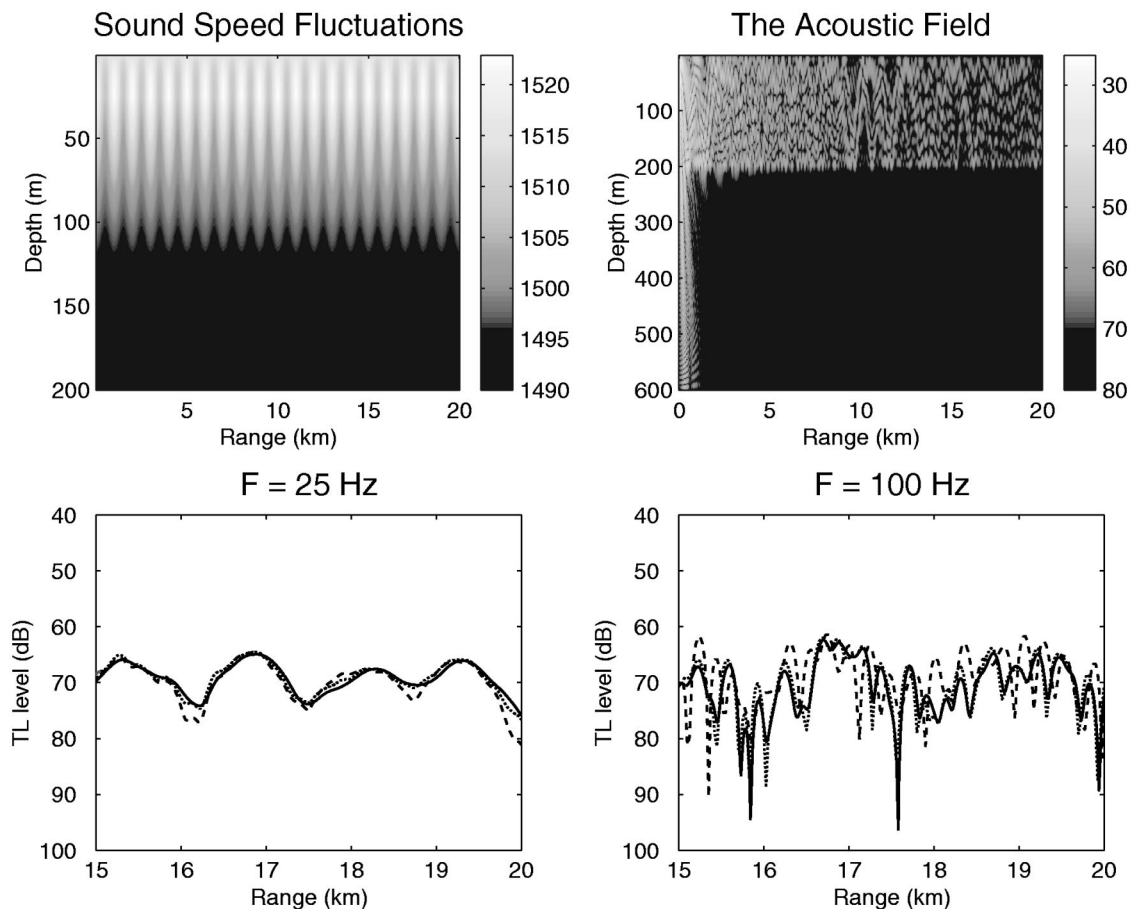
FIG. 2. Propagation through internal waves: The ocean environment consists of a 200 m layer over a bottom half space. The sound speed in the water is modeled to simulate fluctuations due to internal waves. The top left panel shows the sound speed fluctuations in the water. The top right panel shows the acoustic field in this environment due to a 100 Hz source placed at 100 m computed using the coupled mode model. The bottom two panels show a comparison of the transmission between the PE model (solid), the coupled mode model (dotted), and the adiabatic coupled mode model (dashed). The transmission loss is computed at a receiver depth of 100 m and for source frequencies of 25 and 100 Hz.

ergy is conserved among modes. Second, it only contains the modes and their depth derivatives. This means that the coupling matrix can easily be computed using the local modes and their depth derivatives, which can be obtained from any normal mode code such as KRAKEN.[15]

III. EXAMPLES

In this section the one-way coupled mode model developed in the previous section is applied to two range-dependent cases. In the first example we use the one-way coupled model to compute acoustic propagation an oceanic wedge wherein range-dependence is due to variations in water depth. In the second example propagation through an ocean with internal waves is computed where the ocean environment is chosen such that range-dependence is entirely due to variations in sound speed. The results are compared with those obtained using the parabolic equation PE[14] method.

A. Propagation in a wedge

The ocean environment in this example is scaled to correspond to the model tank experiment reported by Coppens and Sanders.[12] The water depth is initially 200 m for the first 5 km and then it slowly decreases to zero in the next 7.5 km

resulting in a wedge angle of approximately 1.55 deg. To approximate the branch cut integral in the modal representation of the field, a 1000 m deep false bottom is used. The waveguide consists of two isovelocity layers: a water layer over a bottom layer. The water sound speed is 1500 m/s and its density is 1.0 g/cm^3. The bottom sound speed is 1700 m/s with a density of 1.15 g/cm^3. The attenuation in the bottom is 0.5 dB/λ. A 25 Hz source is placed at 180 m. These environmental parameters are chosen to correspond to those used by Coppens and Sanders.

The acoustic field in the waveguide is computed using Eq. (2) with the modal coefficients, c_m, obtained from the solution of Eq. (4). The first order differential equation for the modal coefficients, Eq. (4), is solved by using fourth-order Runge–Kutta integration. To obtain the modes and the coupling matrix as a function of range, the wedge is divided into range-independent stair steps. The local modes and the local coupling matrix using Eq. (10) are computed in each stair step and updated in the differential equation. The step size in the two examples that are presented in this paper is 10 m. However, a step size of 50 m gives identical results.

The results of the above computation are shown in Fig. 1. The top left panel in Fig. 1 shows the acoustic field computed using the one-way coupled mode model described in this paper. It can be seen that as the water depth decreases,

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the water modes (there are three water modes in this example) cutoff in the form of discrete beams radiating into the bottom. The experimental data obtained by Coppens and Sanders show the exact same behavior. Jensen and Kuperman[16] used the parabolic equation method to model the acoustic propagation in this example and found results identical to those shown in the top left panel of Fig. 1. They interpreted the slow disappearance of the discrete water modes into the bottom as an indication that energy contained in a given mode does not couple into the next lower mode but couples almost entirely into the continuous mode spectrum. While this effect, which is a consequence of mode coupling, is implicitly accounted for in the parabolic equation (PE) formulation, the coupled mode model explicitly accounts for it through the coupling matrix. The two panels on the right in Fig. 1 show a comparison of the transmission loss computed using the one-way coupled model and the PE model for two receiver depths. The close agreement between the two models clearly demonstrates that the one-way coupled mode model correctly accounts for mode coupling. If the coupling matrix in the one-way coupled mode model are set equal to zero, the one-way coupled mode model reduces to the adiabatic mode model. The bottom left panel in Fig. 1 shows the acoustic field computed using the adiabatic mode solution. Observe that the adiabatic mode solution does not have the correct field behavior near cutoff. While in the coupled mode solution modes gradually radiate their energy to the bottom near cutoff, in the adiabatic mode solution this process occurs abruptly because there is no mechanism for the transfer of energy between modes.

B. Propagation in internal waves

The ocean environment in this example is one of the test cases used at the Shallow Water Acoustic workshop.[13] It consists of 200 m of water over a 400 m, isovelocity bottom. The bottom density 1.5 g/cm^3 and its sound speed was 1700 m/s. The sound speed profile and the velocity fluctuations due to internal waves in the water column are modeled according to[13]

$$c(z,r) = c(z) + \Delta c(z,r),$$

where

$$c(z) = \begin{cases} 1515.0 + 0.016z & z < 26 \\ 1490(1.0 + 0.25(e^{-b} + b - 1.0)) & z > 26 \end{cases}$$

and

$$\Delta c(z,r) = \frac{z}{25} e^{-z/25} \cos 2\pi r.$$

In the above $b = (z - 200)/500$ and r is measured in km. The top left panel in Fig. 2 shows the sound speed profile for this example. The top right panel shows the acoustic field in the waveguide for a 100 Hz source place at 30 m. The bottom two panels in Fig. 2 show a comparison of the adiabatic normal mode and the coupled mode solutions with the PE solution for two source frequencies. In both cases the receiver depth is at 70 m. The coupled mode solution agrees well with PE solution at both frequencies while the adiabatic

normal mode solution does not agree with the parabolic equation solution at all. This is more evident at the higher frequency where the effects of the internal waves, and thus the mode coupling due to them, is stronger.

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APPENDIX A: THE MODE ORTHOGONALITY RELATIONSHIP

The mode orthogonality equation is obtained by multiplying the mode equation (6) by p_m and Eq. (7) by p_n subtracting the resulting equations and integrating to give

$$\int_0^B \left[p_m \, \partial_z \left(\frac{1}{\rho} \partial_z p_n \right) - p_n \, \partial_z \left(\frac{1}{\rho} \partial_z p_m \right) \right] dz$$

$$+ (k_m^2 - k_n^2) \int_0^B \frac{1}{\rho} p_m p_n \, dz = 0.$$

Integrating the first integral by parts results in boundary terms

$$p_m \frac{1}{\rho} \partial_z p_n \bigg|_0^B - p_n \frac{1}{\rho} \partial_z p_m \bigg|_0^B.$$

Since either the mode or its derivative is zero at the boundaries, there is no contribution from the above interface terms. What remains is

$$(k_m - k_n)(k_m + k_n) \int_0^B \frac{1}{\rho} p_m p_n \, dz = 0.$$

For $m \neq n$,

$$(k_m + k_n) \int_0^B \frac{1}{\rho} p_m p_n \, dz = 0,$$

and for $m = n$ we choose to normalize the modes such that

$$(k_m + k_n) \int_0^B \frac{1}{\rho} p_m p_n \, dz = 2 k_n \delta_{mn}.$$

APPENDIX B: DETAILS LEADING TO EQ. (4)

We start by multiplying the first equation in Eq. (1) by u_m and the second equation in Eq. (1) by p_m, adding the two equations and integrating to get

$$i k_n \int_0^B (p_n u_m + u_n p_m) dz$$

$$= \int_0^B \left(-\frac{p_n p_m}{\rho c^2} - \partial_z \left(\frac{1}{\rho \omega^2} \partial_z p_n \right) p_m + \rho \omega^2 u_n u_m \right) dz.$$

Integrating the middle term on the right-hand side by parts yields

$$\int_0^B \partial_z \left(\frac{1}{\rho c^2} \partial_z p_n \right) p_m \, dz = \left[\left(\frac{1}{\rho \omega^2} \partial_z p_n \right) p_m \right]_-^+$$

$$- \int_0^B \left(\frac{1}{\rho \omega^2} \partial_z p_n \right) \partial_z p_m \, dz.$$

Since the quantities inside the square brackets are continuous across the interface, the boundary term is zero. This results in

$$ik_n \int_0^B (p_n u_m + u_n p_m) dz = \int_0^B \left(-\frac{p_n p_m}{\rho c^2} + \frac{1}{\rho \omega^2} (\partial_z p_n) \right.$$

$$\left. \times (\partial_z p_m) + \rho \omega^2 u_n u_m \right) dz. \quad \text{(B1)}$$

Next consider the following term in Eq. (1), which can be integrated to give,

$$\int_0^B \hat{\mathbf{z}} \cdot (\partial_z \mathbf{u}_n) p_m \, dz = [\hat{\mathbf{z}} \cdot \mathbf{u}_n p_m]_-^+ - \int_0^B (\hat{\mathbf{z}} \cdot \mathbf{u}_n \, \partial_z p_m) dz. \quad \text{(B2)}$$

The continuity condition for the normal component of the displacement can be written as

$$[\hat{\mathbf{n}} \cdot \mathbf{u}_n]_-^+ = [(-\hat{\mathbf{z}} + \partial_x h \hat{\mathbf{x}}) \cdot \mathbf{u}_n]_-^+ = 0.$$

This gives,

$$[\hat{\mathbf{z}} \cdot \mathbf{u}_n]_-^+ = (\partial_x h) u_n.$$

Since p_m is continuous across the interface, Eq. (B2) reduces to

$$\int_0^B \hat{\mathbf{z}} \cdot (\partial_z \mathbf{u}_n) p_m \, dz = [(\partial_x h) u_n p_m]_-^+ - \int_0^B (\hat{\mathbf{z}} \cdot \mathbf{u}_n \, \partial_z p_m) dz.$$

Substituting this into Eq. (3) and using Eq. (B1) gives

$$\sum_n \left\{ \partial_x c_n \int_0^B (u_n p_m + p_n u_m) dz + c_n \int_0^B (p_m \, \partial_x u_n \right.$$

$$+ u_m \, \partial_x p_n) dz + c_n [(\partial_x h) u_n p_m]_-^+ e^{i(k_n - k_m)x} = 0.$$

Next substituting for $u_n = i k_n p_n / \omega^2 \rho$ gives

$$\sum_n \left\{ (\partial_x c_n)(k_n + k_m) \int_0^B \frac{1}{\rho} p_n p_m \, dz + c_n (k_n + k_m) \right.$$

$$\times \int_0^B (\partial_x p_n) \frac{p_m}{\rho} dz + c_n (\partial_x k_n) \int_0^B \frac{1}{\rho} p_n p_m \, dz$$

$$+ c_n k_n \int_0^B \partial_x \left(\frac{1}{\rho} \right) p_n p_m dz$$

$$+ c_n \left[(\partial_x h) \frac{k_n p_n p_m}{\rho} \right]_-^+ e^{i(k_n - k_m)x} \right\} = 0. \quad \text{(B3)}$$

Using the orthogonality of the modes we find,

$$2 \partial_x c_m k_m + c_m \, \partial_x k_m = \sum_{n \neq m} A_{mn} c_n,$$

where the coupling matrix, A_{mn} is defined by Eq. (10). For $m = n$ Eq. (B3) becomes,

$$2 \partial_x c_n k_n + c_n \, \partial_x k_n + c_n k_n \int_0^B \left(2(\partial_x p_n) \frac{p_n}{\rho} + p_n^2 \, \partial_x \left(\frac{1}{\rho} \right) \right) dz$$

$$+ c_n \left[(\partial_x h) \frac{k_n p_n^2}{\rho} \right]_-^+ = 0.$$

This can be written as

$$2 \partial_x c_n k_n + c_n \, \partial_x k_n + c_n k_n \int_0^B \partial_x \left(\frac{p_n^2}{\rho} \right) dz$$

$$+ c_n \left[(\partial_x h) \frac{k_n p_n^2}{\rho} \right]_-^+ = 0.$$

It is shown in Appendix C that

$$\int_0^B \partial_x \left(\frac{p_n^2}{\rho} \right) dz = -\partial_x h \left[\frac{1}{\rho} p_n^2 \right]_-^+, \quad \text{(B4)}$$

which gives

$$2 \partial_x c_n k_n + c_n \, \partial_x k_n = 0.$$

APPENDIX C: DERIVATION OF EQ. (B4)

The integral across the waveguide can be written as,

$$\int_0^B \partial_x \left(\frac{p_n^2}{\rho} \right) dz = \int_0^{h^+} \partial_x \left(\frac{p_n^2}{\rho} \right) dz + \int_{h^-}^B \partial_x \left(\frac{p_n^2}{\rho} \right) dz.$$

Each one of the above integrals can be written as,

$$\int_0^{h^+} \partial_x \left(\frac{p_n^2}{\rho} \right) dz = \partial_x \int_0^{h^+} \left(\frac{p_n^2}{\rho} \right) dz - \partial_x (h^+) \left[\frac{p_n^2}{\rho} \right]_{h^+},$$

and

$$\int_{h^-}^B \partial_x \left(\frac{p_n^2}{\rho} \right) dz = \partial_x \int_{h^-}^B \left(\frac{p_n^2}{\rho} \right) dz + \partial_x (h^-) \left[\frac{p_n^2}{\rho} \right]_{h^-}.$$

Substituting for these quantities we find

$$\int_0^B \partial_x \left(\frac{p_n^2}{\rho} \right) dz = \partial_x \int_0^B \left(\frac{p_n^2}{\rho} \right) dz - \partial_x h \left[\frac{p_n^2}{\rho} \right]_-^+.$$

The integral on the right-hand side is a constant which gives

$$\int_0^B \partial_x \left(\frac{p_n^2}{\rho} \right) dz = -\partial_x h \left[\frac{p_n^2}{\rho} \right]_-^+.$$

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