Acoustic Modelling in Range-Dependent Environments

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Abstract

Two fundamental issues related to bottom-interacting PE modelling in environments with strong bathymetric changes are addressed. First, it is demonstrated that the classical problem of angular limitations in PE's can be virtually eliminated by solving one of the higher-order forms recently proposed. Secondly, it is shown that the standard stair-step representation of a sloping bottom may result in significant prediction errors. In fact, current PE implementations are <u>not</u> energy conserving. The problem is shown to derive from the approximate treatment of the interface conditions at vertical boundaries along the stair steps. Several improved interface conditions are proposed.

1 Introduction

Among the various modelling techniques available for solving propagation problems in range-dependent environments, the parabolic equation technique is without doubt the most popular[1,2]. This technique was originally introduced by Tappert[1] in a narrow-angle form solvable in a range-marching scheme based on Fourier transforms. Improved PE's with better angular coverage were soon derived and implemented in numerical schemes based on finite differences and finite elements[3]. However, the search for PE forms which provide accurate solutions over the entire angular spectrum has continued, and some recent developments indicate that the issue has finally been solved. This could lead one to believe that PE modelling has now reached such a level of perfection that accurate field solutions can be obtained for even the most complex environmental situations.

Considering the above, it definitely came as a surprise to modellers when it was recently discovered that current PE implementations are not energy conserving in sloping bottom environments[4]. In fact, prediction errors of several decibels may occur for bottom slopes of a few degrees. Interesting enough, this problem, which affects mean level predictions, had been entirely overlooked, while much effort over the years went into correcting phase errors, which have no effect on mean levels.

We first review 15 years of progress in developing a true wide-angle PE capability, and then turn to the important new problem of energy conservation in sloping bottom environments.

2 PE Approximations and Angular Limitations

2.1 Derivation of Parabolic Approximations

The starting point is the Helmholtz equation for a constant-density medium in cylindrical coordinates (r, z, ϑ) and for a harmonic point source of time dependence $\exp(-i\omega t)$:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} + k_0^2 n^2 p = 0, \tag{1}$$

where we have assumed azimuthal symmetry and hence no dependence on the ϑ coordinate. Here $p(r, z)$ is the acoustic pressure, $k_0 = \omega / c_0$ is a reference wavenumber, and $n(r, z) = c_0/c(r, z)$ is the index of refraction. It is convenient next to remove cylindrical spreading via the transformation

$$p(r, z) = r^{-1/2}\psi(r, z), \tag{2}$$

from which it follows that $\psi(r, z)$ satisfies the farfield $(k_0 r \gg 1)$ Helmholtz equation

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2 n^2 \psi = 0. \tag{3}$$

Introducing the operator notation

$$P = \frac{\partial}{\partial r}, \quad Q = \sqrt{n^2 + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}}, \tag{4}$$

Eq. (3) can be written in the concise form

$$\left(P^2 + k_0^2 Q^2\right)\psi = 0. \tag{5}$$

The next step is to factor this equation into two components, an outgoing and an incoming wave component, according to:

$$(P - ik_0 Q)(P + ik_0 Q)\psi - ik_0[P, Q]\psi = 0, \tag{6}$$

where $[P, Q]\psi = PQ\psi - QP\psi$ is the commutator of the operators P and Q. For range-independent media where $n \equiv n(z)$, the two operators commute and the last term in Eq. (6) is equal to zero. Selecting only the outgoing wave component we then obtain:

$$P\psi = ik_0 Q\psi$$

or

$$\frac{\partial \psi}{\partial r} = ik_0\left(\sqrt{n^2 + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}}\right)\psi. \tag{7}$$

This equation is a genuine one-way wave equation, which for range-independent environments is exact within the limits of the farfield approximation. The equation is evolutionary in range and provides the basis for obtaining various forms of parabolic approximations, i.e. partial differential equations in first order with respect to r. These forms will follow as a result of approximations to the pseudo-differential operator Q, whose properties preclude the solution of Eq. (7) itself.

2.2 Expansion of square-root operator

For convenience we write the square-root operator Q given by Eq. (4) as

$$Q = \sqrt{1+q}, \qquad q = n^2 - 1 + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}. \tag{8}$$

A solvable parabolic wave equation is now obtained by introducing some kind of functional approximation to the square-root operator. An excellent overview of several different families of approximants has been given by Halpern and Trefethen[5]. We shall here present just the two most common forms. One is a rational-function approximation of the form

$$\sqrt{1+q} \approx \frac{a_0 + a_1 q}{b_0 + b_1 q}, \tag{9}$$

where the coefficients can be chosen so as obtain three well-known parabolic approximations due to Tappert[1], Claerbout[6] and Greene[7].

The second family is based on a Padé series approximation of the form [8]:

$$\sqrt{1+q} \approx 1 + \sum_{j=1}^{m} \frac{a_{j,m} q}{1 + b_{j,m} q} + O(q^{2m+1}), \tag{10}$$

where m is the number of terms in the Padé expansion and

$$a_{j,m} = \frac{2}{2m+1}\sin^2\left(\frac{j\pi}{2m+1}\right), \qquad b_{j,m} = \cos^2\left(\frac{j\pi}{2m+1}\right).$$

We shall shortly demonstrate that very-wide-angle PE's can be obtained by retaining several terms from this expansion.

2.3 Phase Errors and Angular Limitations

We proceed to write down a series of functional approximations of the operator Q, which have been used extensively in the underwater acoustics community. We start with the exact form from the Helmholtz equation

$$Q = \sqrt{1+q}. \qquad [Helmholtz] \tag{11}$$

Next we choose $a_0 = 1$, $a_1 = 0.5$, $b_0 = 1$, and $b_1 = 0$ in Eq. (9) to obtain the approximate form

$$Q_1 = 1 + \frac{q}{2}, \qquad [Tappert], \tag{12}$$

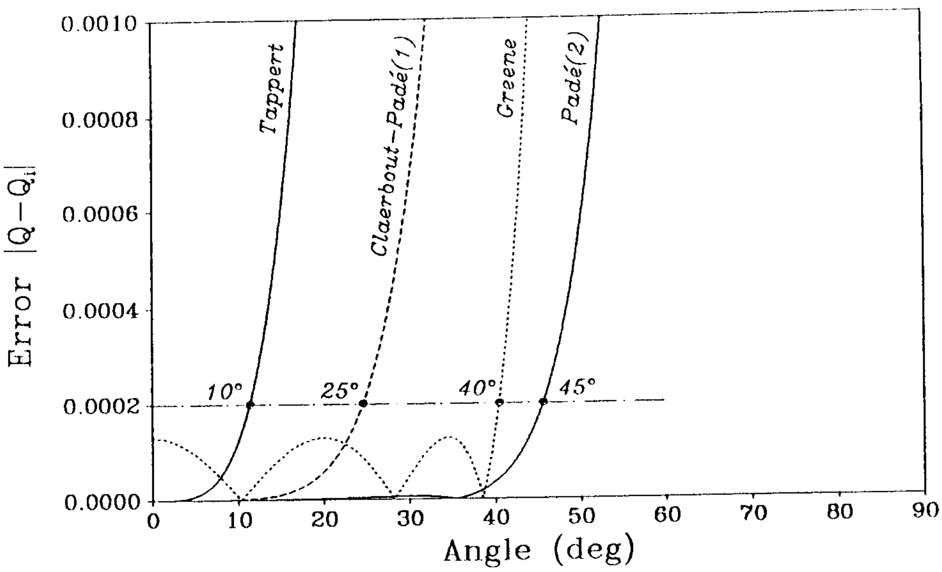

Figure 1: Phase errors vs propagation angle for different parabolic equation approximations.

which leads to the standard parabolic equation proposed by Tappert [1].

A second important Q-approximation is obtained by selecting $a_0 = 1$, $a_1 = 0.75$, $b_0 = 1$, and $b_1 = 0.25$:

$$Q_2 = \frac{1 + 0.75q}{1 + 0.25q}, \qquad [\textit{Claerbout, Padé}\,(1)] \tag{13}$$

which is the wide-angle PE due to Claerbout[6]. This form is also obtained by retaining only one term in the Padé expansion of Eq. (10).

A third rational-function approximation due to Greene[7] has slightly different coefficients:

$$Q_3 = \frac{0.99987 + 0.79624q}{1 + 0.30102q}. \qquad [\textit{Greene}] \tag{14}$$

This form was obtained by minimizing phase errors over the angle interval $0 - 40^o$.

Finally we present the form obtained by including two terms in the Padé expansion:

$$Q_4 = 1 + \frac{0.138q}{1 + 0.655q} + \frac{0.362q}{1 + 0.096q}. \qquad [\textit{Padé}\,(2)] \tag{15}$$

An estimate of phase errors as a function of propagation angle can be obtained by considering a plane-wave component in a homogeneous medium given by $\psi = \exp[ik_0(r\,\cos\theta \pm z\,\sin\theta)]$. It then follows form Eq. (8) that $q = -\sin^2\theta$.

The phase errors for the various PE approximations defined as $\mid Q - Q_i \mid$ are displayed graphically as a function of angle in Fig. 1. As indicated by the horizontal dashed line, we have arbitrarily chosen the acceptable error to be 0.0002, which is seen to result in the following approximate angle limitations: 10^o (Tappert), 25^o (Claerbout, Padé (1)), 40^o (Greene), and 45^o (Padé (2)). We also notice that while the

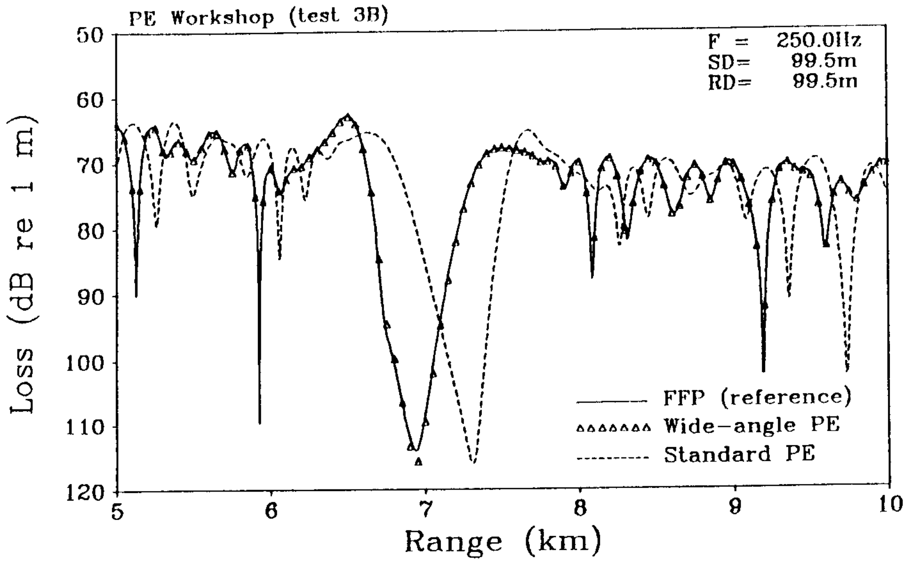

Figure 2: Comparison of transmission loss results for narrow and wide-angle PE's with an FFP reference solution.

phase error increases with angle for the Tappert and Claerbout equations, Greene's 40^o expansion has errors of equal magnitude over the entire range from 0 to 40^o. However, the best PE approximation is clearly the Padé(2) form, which has very small phase errors for angles within $\pm 40^o$ of the main propagation direction. As shown by Collins[8], this wide-angle capability can be extended to nearly $\pm 90^o$ by including more terms in the Padé expansion given by Eq. (10).

A classical example illustrating the effects of phase errors on computed transmission loss is shown in Fig. 2. This test problem was first presented at the PE Workshop in 1981[2]. We consider a Pekeris waveguide of depth 100 m with both source and receiver placed near the bottom at 99.5 m. The water speed is 1500 m/s and the bottom speed 1590 m/s, resulting in a critical angle of 19.4°. The density ratio between water and bottom is 1.2 and the bottom attenuation is 0.5 dB/λ. At a frequency of 250 Hz there are 11 propagating modes with propagation angles between 1.7° and 18.5°. Considering the phase error results given in Fig. 1, it is not surprising that the result from the standard PE (Tappert eq.) matches poorly the FFP reference solution in Fig. 2. On the other hand, the wide-angle Claerbout equation is seen to provide accurate results for this test problem. For completeness we mention that the two PE results were obtained using a modal source.

In summary, we have shown that the various parabolic approximations to the Helmholtz equation all have inherent phase errors, which limit their applicability to a certain range of angles around the main propagation direction. However, recent developments based on Padé series expansions seem virtually to have eliminated the small-angle restriction generally associated with parabolic wave equations. Of course, the wide-angle capability is not achieved without additional computational effort.

3 The Problem of Energy Conservation in PE's

Since the early 1970's, when the first parabolic equation solution based on Fourier transforms appeared in the underwater acoustics community [1], the PE technique has been applied extensively to model propagation in range-dependent ocean environments with strong bathymetric changes. In fact, the rational behind the continual search for more wide-angle PE forms was to obtain an accurate treatment of bottom-interacting propagation, which often involves steep propagation angles (critical reflection angles of $20 - 30^o$ are typical for sandy seafloors). With phase errors practically eliminated from the wide-angle PE codes, these were subsequently used with confidence to model propagation in sloping bottom situations, both on continental shelfs and over seamounts. There is, however, a fundamental problem of energy conservation in current PE implementations of sloping interfaces, a problem which was recognized just recently and which may result in prediction errors of several decibels for moderate bottom slopes.

3.1 ASA Benchmark Results

Examples of inaccurate PE results for upslope propagation in wedge-shaped oceans first appeared among the range-dependent benchmark solutions solicited by the Acoustical Society of America in 1987[4]. Figure 3 shows one of these benchmark results for a 2.86^o wedge. The initial water depth is 200 m decreasing linearly to 0 at a range of 4 km. The source depth is 100 m and the source frequency 25 Hz. The water sound speed is 1500 m/s, while the bottom speed is 1700 m/s. The density ratio between water and bottom is 1.5 and the bottom attenuation is 0.5 dB/λ. The reference solution given by the solid line was obtained from a full-spectrum two-way coupled-mode solution[9]. On the same graph is displayed the one-way COUPLE result which was found to be identical to the result obtained from a finite-difference implementation of the Claerbout PE. The key point is that the two one-way solutions have a 2 dB lower level at longer ranges. This difference was initially thought to be due to the neglect of backscattering in the one-way solutions, but it was subsequently realized that the backscattered field in this case is negligible. In fact, as shown in Fig. 3(b) (note change of dB scale), the strength of the backscattered field component extracted from the two-way COUPLE result, is 40-50 dB lower than the outgoing field component.

Since the solution is entirely outgoing, then why are the one-way results in error? This can be easily understood by looking at the consequences of the use of a stair-step approximation to a sloping bottom. Figure 4 displays a few vertical segments separating media of different properties. The important interfaces with strong impedance contrasts are the horizontal interfaces I_h and the vertical interfaces I_v along the stair-steps. While boundary conditions at horizontal interfaces (continuity of pressure and vertical particle velocity) are accurately implemented, the vertical boundaries I_v are treated very losely. In fact, an *a priori* assumption of the solution being outgoing only, permits just one vertical boundary condition to be satisfied. When solving

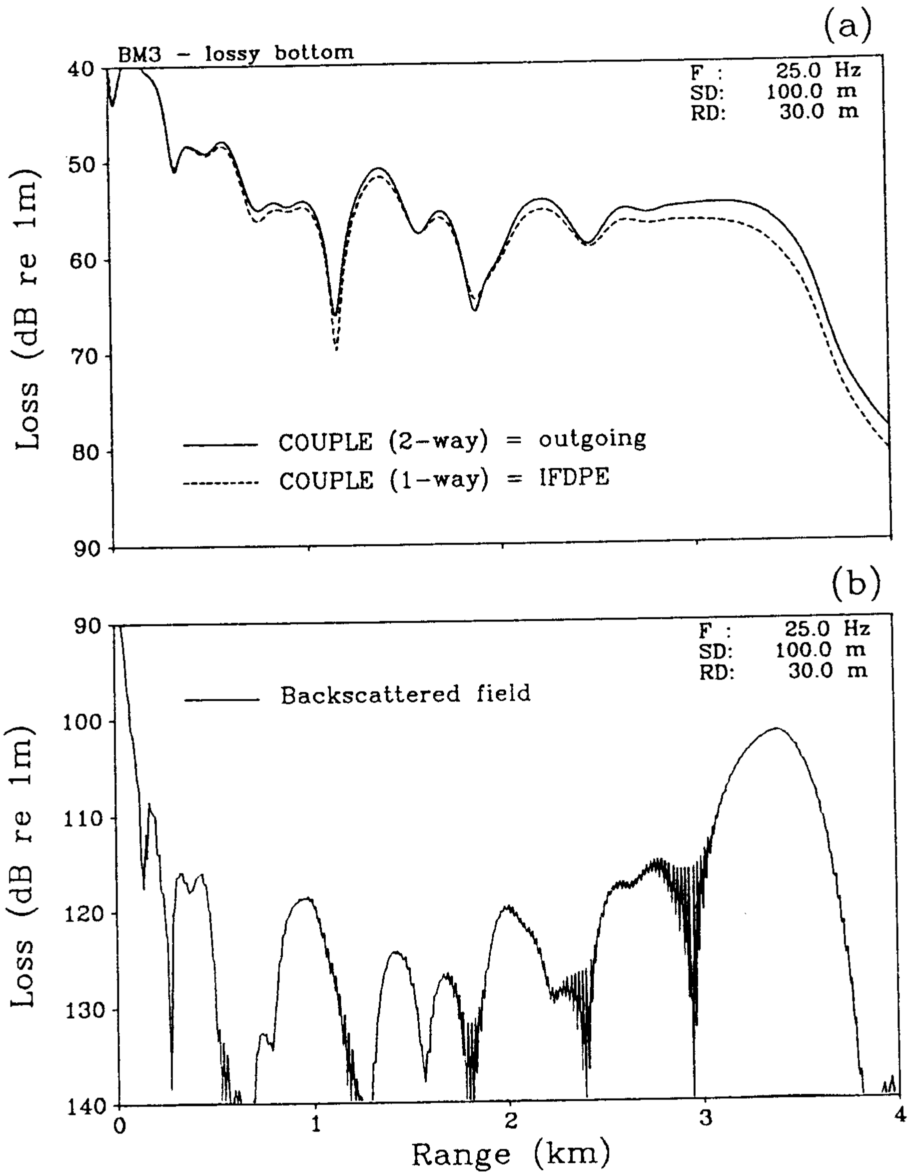

Figure 3: Coupled-mode results for 2.86° wedge with lossy penetrable bottom. (a) two-way and outgoing field components, (b) backscattered component.

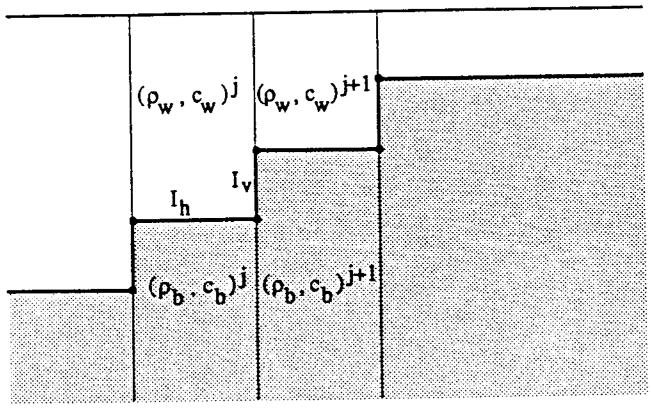

Figure 4: Stair-step representation of sloping interface.

for pressure in a finite-difference PE implementation, the boundary condition being satisfied is continuity of pressure across vertical interfaces. Since split-step Fourier implementations generally solve for a density-reduced pressure $p/\sqrt{\rho}$, these codes provide continuity of reduced pressure across vertical interfaces. It is clear, however, that the full boundary condition cannot be satisfied within the framework of a one-way solution.

3.2 Test of Interface Conditions

We next address the question of which type of approximate boundary condition should be used in one-way solutions in order to improve accuracy. Some guidance can be gained by examining the question in the context of a 1-D wave equation. Thus, we consider a problem with $c(x)$ being the sound speed and $\rho(x)$ being the density as a function of axial distance x.

In our discretization, the medium is approximated by a sequence of segments within which it is assumed that both sound speed and density is constant. The solution therefore assumes the form:

$$p(x) = A_j e^{ik_j(x-x_j)} + B_j e^{-ik_j(x-x_j)}, \tag{16}$$

where $[x_j, x_{j+1}]$ are the endpoints of the jth segment and k_j is the wavenumber in that segment. With subscript x indicating differentiation with respect to range, we consider four possible interface conditions: 1) continuity of energy flux pp_x/ρ; 2) continuity of pressure p; 3) continuity of particle velocity p_x/ρ; 4) continuity of reduced pressure $p/\sqrt{\rho}$.

3.2.1 Continuity of Energy Flux

The energy flux is given by $E = pp_x/\rho$. We assume that $B_j = 0$ for each segment so that there is no backscattered field. By matching pp_x/ρ we conserve energy flux.

Thus

$$A_{j+1}^2 = A_j^2 \frac{k_j}{k_{j+1}} \frac{\rho_{j+1}}{\rho_j} e^{i2k_j(x_j - x_{j-1})}, \tag{17}$$

$$\Rightarrow A_{j+1} = A_0 \prod_{m=0}^{j} \sqrt{\frac{k_{m-1}}{k_m} \frac{\rho_m}{\rho_{m-1}}} e^{ik_m(x_m - x_{m-1})}, \tag{18}$$

$$= A_0 \sqrt{\frac{k_0}{k_j} \frac{\rho_{j+1}}{\rho_0}} e^{i \sum_{m=0}^{j} k_m (x_m - x_{m-1})}. \tag{19}$$

Now taking the limit as the number of segments goes to infinity we obtain:

$$p(x) = A_0 \sqrt{\frac{k_0}{k(x)} \frac{\rho(x)}{\rho_0}} e^{i \int_0^x k(s)\, ds}. \tag{20}$$

Note that for energy to be conserved the wave amplitude has a square-root dependence on the change in impedance ρc along the propagation path.

As an alternative we could impose the full boundary condition (continuity of pressure and horizontal particle velocity) across individual interfaces in the context of a marching solution. This single scatter result implies that the backscattered field component is used only in the interface matching and subsequently ignored. It is not difficult to show that the single scatter result is identical to the one given in Eq. (20), and hence is energy conserving in the forward sense.

3.2.2 Continuity of Pressure

We again assume that $B_j = 0$ for each segment so that there is no backscattered field. Matching pressure we obtain without difficulty

$$p(x) = A_0 e^{i \int_0^x k(s)\, ds}. \tag{21}$$

Notice that the phase term is identical to the result of Eq. (20) however the amplitude factor is wrong. The pressure-matched solution clearly results in a loss of energy when passing from a low-impedance to a high-impedance medium. This observation is consistent with the numerical results in Fig. 3.

3.2.3 Continuity of Particle Velocity

By matching p_x/ρ at each interface we obtain,

$$p(x) = A_0 \frac{k_0}{k(x)} \frac{\rho(x)}{\rho_0} e^{i \int_0^x k(s)\, ds}. \tag{22}$$

Once again, the phase factor is correct and the amplitude incorrect. The energy conservation term in Eq. (20) is the geometric mean of the pressure-matched and velocity-matched solutions.

3.2.4 Continuity of Reduced Pressure

Finally, matching $p/\sqrt{\rho}$ at each interface leads to:

$$p(x) = A_0\sqrt{\frac{\rho(x)}{\rho_0}}e^{i\int_0^x k(s)\,ds}. \tag{23}$$

Thus by matching reduced pressure we correct for the errors due to density variation but not for those due to the change in sound speed.

In summary, the solution which matches pressure alone show serious deficiencies for moderate density variation. Matching particle velocity is also a poor choice, however, matching reduced pressure corrects entirely for the density effect. If $c(x)$ varies much less than $\rho(x)$ the reduced pressure condition would correct most of the error. A further improvement may be obtained using the single-scatter approximation. In the next section we show the results of implementing different boundary conditions in a one-way coupled-mode solution.

3.3 One-Way Coupled-Mode Results

Since the problem of energy conservation in PE's is accentuated by increasing the bottom slope, we have here selected a somewhat extreme situation where the wedge-shaped ocean has a slope angle of 12.7°. The initial water depth is 1000 m decreasing linearly to 100 m at a range of 4 km. The source depth is 50 m and the source frequency 25 Hz. The water sound speed is 1500 m/s while the bottom speed is 1700 m/s. The density ratio between water and bottom is 2.0 and the bottom attenuation is 0.5 dB/λ. The reference solution was again obtained from a full-spectrum two-way coupled-mode solution[9].

One-way coupled-mode results for various interface conditions are given in Fig. 5. Note that this problem has been solved for a line source in plane geometry. The upper graph shows transmission loss over depth for upslope propagation (deep to shallow) while the lower graph gives equivalent results for downslope propagation (shallow to deep). The reference solution is the full line representing both the two-way coupled-mode result and the single-scatter (continuity of p and u) one-way result. We again find that the backscattered field is insignificant and that the solution is fundamentally outgoing. This, in turn, explains why the single-scatter solution agrees almost perfectly with the full two-way result.

Next we note that the standard pressure matching inherent in most finite-difference and finite-element PE's results in level errors of 5-7 dB, with energy loss for upslope propagation and energy gain for downslope propagation. These results are in qualitative agreement with the indications obtained from the 1-D analysis in the previous section. If we now turn to the $p/\sqrt{\rho}$ matching as implemented in most split-step PE's, we see that the error is reduced by approximately 75%. Considering that this test problem represents an extreme case, we may conclude that the $p/\sqrt{\rho}$ is adequate for most practical problems in underwater acoustics. However, the pressure matching currently implemented in finite-difference PE's is definitely not adequate.

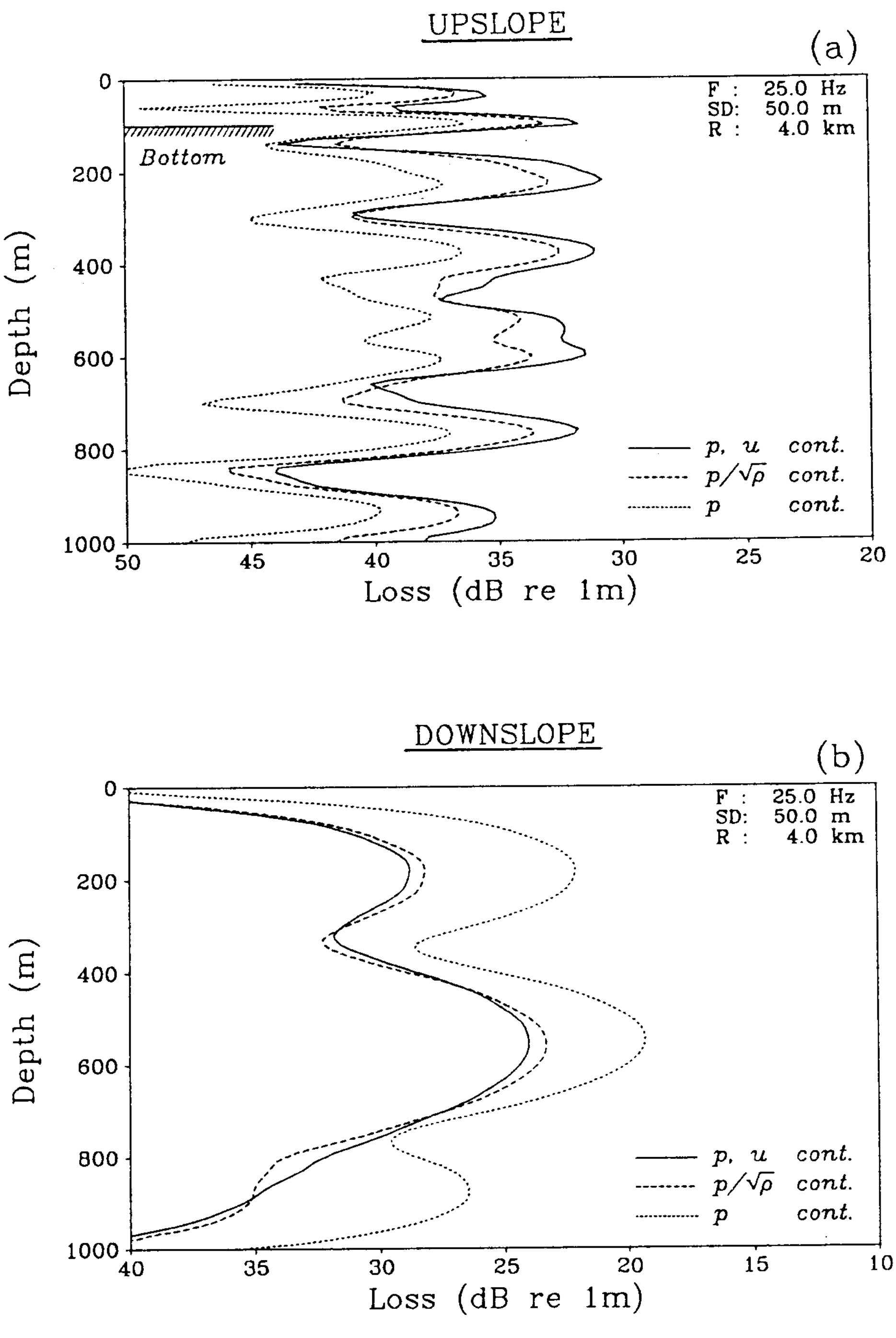

Figure 5: Coupled-mode results for 12.7° wedge with strong impedance contrast at the water/bottom interface. (a) One-way results for upslope propagation with different boundary conditions at vertical interfaces, (b) equivalent results for downslope propagation.

implement directly a sloping-bottom boundary
McDaniel[10] and more recently by Huang[11]. However, numerical evidence of the accuracy of this type of interface representation is currently not available.

4 Summary and Conclusions

Two problems associated with parabolic equation modelling have been addressed. One is the angular limitations caused by inherent phase errors in PE approximations. We have demonstrated that this problem after more than ten years of research finally has been solved. Various high-order PE forms have been derived which, with additional computational effort, permit accurate field calculations to nearly $\pm 90^\circ$. The second problem is associated with the stair-step modelling of sloping bottoms. We have shown that one-way PE formulations treat vertical interfaces in an approximate fashion leading to energy loss for upslope propagation and energy gain for downslope propagation. While improved interface conditions have been successfully tested in a one-way coupled-mode code, the appropriate remedy for existing PE codes still needs to be worked out.

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